Efficient thermodynamic description of the Gaudin-Yang model

Ovidiu I. Pâţu and Andreas Klümper

Institute for Space Sciences, Bucharest and Bergische Universität Wuppertal

Outline

1. The Gaudin-Yang model
2. The Gaudin-Yang model as the continuum limit of the Perk-Schultz spin chain
3. Alternative thermodynamic description. Results
4. Numerical results for the contact
5. Conclusions and future plans

Motivation

Experimental realization of 1D integrable models

- Lieb-Liniger model: T. Kinoshita et al., *Observation of a One-Dimensional Tonks-Girardeau Gas,* Science **305**, 1125 (2004); B. Paredes et al., *Tonks-Girardeau gas of ultracold atoms in an optical lattice,* Nature **429**, 277 (2004).
- Yang-Yang thermodynamics for the Lieb-Liniger model: A.H. van Amerongen et al., *Yang-Yang thermodynamics on an atom chip*, Phys. Rev. Lett. **100**, 090402 (2008).
- Atractive fermionic Gaudin-Yang model ($N = 2$): Y. Liao et. al., *Spin-imbalance in a one-dimensional Fermi gas*, Nature **467**, 567 (2010).
- Repulsive fermionic Gaudin-Yang model ($N = 2 \cdots 6$): G. Pagano et al., *A one-dimensional liquid of fermions with tunable spin*, Nature Physics **10**, 198 (2014).

The Gaudin-Yang model

The Hamiltonian (Gaudin, Yang 1967, Sutherland 1968)

$$\mathcal{H}_{GY} = \frac{\hbar^2}{2m}\sum_{\sigma=\uparrow,\downarrow} \partial_x\Psi_\sigma^\dagger(x)\partial_x\Psi_\sigma(x) + c\sum_{\sigma,\sigma'=\uparrow,\downarrow}\Psi_\sigma^\dagger(x)\Psi_{\sigma'}^\dagger(x)\Psi_{\sigma'}(x)\Psi_\sigma(x) - \sum_{\sigma=\uparrow,\downarrow}\mu_\sigma\Psi_\sigma^\dagger(x)\Psi_\sigma(x)$$

$c > 0$ coupling constant, μ_σ chemical potentials, m the mass of the particles.

- Fermionic model

$$\{\Psi_\sigma(x), \Psi_{\sigma'}^\dagger(y)\} = \delta_{\sigma,\sigma'}\delta(x-y)$$

- Bosonic model

$$[\Psi_\sigma(x), \Psi_{\sigma'}^\dagger(y)] = \delta_{\sigma,\sigma'}\delta(x-y)$$

Hamiltonian in first quantization

$$\mathcal{H}_{GY} = -\sum_{i=1}^{M}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + 2c\sum_{1\le i<j\le M}\delta(x_i-x_j) - \sum_{\sigma=\uparrow,\downarrow}\mu_\sigma M_\sigma$$

$M = M_\uparrow + M_\downarrow$ total number of particles. We introduce $\mu = (\mu_\uparrow + \mu_\downarrow)/2$ $H = (\mu_\uparrow - \mu_\downarrow)/2$
$\gamma = c/n\,, \tau = T/n^2$ (n=total density, T=temperature) ($\hbar = 2m = k_B = 1$)
Strong interactions: $\gamma \gg 1$, Weak interactions: $\gamma \ll 1$

Thermodynamic Bethe Ansatz: Fermionic system

Grandcanonical potential (Takahashi 1971, Lai 1971,1973) ($\beta = 1/T$)

$$\phi_{TBA}(\beta, \mu, H) = -\frac{1}{2\pi\beta}\int_{-\infty}^{+\infty} dk\, \ln(1+\zeta(k))$$

with $\zeta(k)$ satisfying the TBA equations

$$\begin{aligned}
\ln\zeta(k) &= -\beta(k^2-\mu) + R * \ln(1+\zeta(k)) + f * \ln(1+\eta_1(k))\,,\\
\ln\eta_1(k) &= f * (\ln(1+\eta_2(k)) - \ln(1+\zeta(k)))\,,\\
\ln\eta_n(k) &= f * (\ln(1+\eta_{n-1}(k)) + \ln(1+\eta_{n+1}(k)))\,,\quad n = 2,\cdots,\infty\,,\\
\lim_{n\to\infty}\frac{\ln\eta_n(k)}{n} &= 2\beta H
\end{aligned}$$

$g * h(k) \equiv \int_{-\infty}^{+\infty} g(k-k')h(k')\,dk'$, $f(k) = 1/[2c\cosh(\pi k/c)]$, $R(k) = b_1 * f(k)$ with $b_1(k) = c/[2\pi((c/2)^2 + k^2)]$

- Groundstate is unpolarized (Lieb-Mattis 1962).
- Groundstate properties and low-lying excitations (Schlottman 1993,1994)

Thermodynamic Bethe Ansatz: Bosonic system

Grandcanonical potential (Gu, Li, Ying, Zhao 2002)

$$\phi(\mu, H, \beta) = -\frac{1}{2\pi\beta}\int_{-\infty}^{+\infty} dk\, \ln(1+\eta_1(k))\,,$$

with $\eta_1(k)$ satisfying the TBA equations

$$\ln\eta_1(k) = -\beta(k^2-\mu-H) + a_3 * f * \ln(1+\eta_1(k)) + f * \ln(1+\eta_2(k))\,,$$
$$\ln\eta_n(k) = f * (\ln(1+\eta_{n-1}(k)) + \ln(1+\eta_{n+1}(k)))\,,\quad n = 2, \cdots, \infty\,,$$
$$\lim_{n\to\infty}\frac{\ln\eta_n(k)}{n} = 2\beta H.$$

$g * h(k) \equiv \int_{-\infty}^{+\infty} g(k-k')h(k')\, dk'$, $f(k) = 1/[2c\cosh(\pi k/c)]$ and $a_n(k) = nc/[2\pi((nc/2)^2+k^2)]$.

- Groundstate is polarized (Eisenberg, Lieb 2002, Yang, Li 2003, Suto 1993)
- Groundstate properties and low-lying excitations (Gu, Li, Ying, Eckern 2003)

Numerical implementation of the TBA

- Requires the truncation of the system at a certain level *nmax*. For $n > nmax$ replace the functions by their asymptotic values.
- In order to obtain accurate results for intermediate and high temperatures *nmax* needs to be large which makes the entire procedure rather cumbersome.
- Fermionic system: Caux, Klauser, van der Brink 2009; Klauser Caux 2011

- Start from the TBA equations and derive more manageable expressions in various limits
- Low-temperature strong coupling: Guan, Batchelor, Takahashi 2007; Lee, Guan, Sakai, Batchelor 2012

Is there a more efficient way of computing the thermodynamic properties ?

Quantum Transfer Matrix thermodynamics

Integrable lattice models at $T > 0$: Quantum Transfer Matrix produces a finite number of NLIEs for the thermodynamics (M. Suzuki 1985; T. Koma 1987; M. Suzuki and M. Inoue 1987; A. Klümper 1992, 1993)

- Largest eigenvalue of QTM $\rightarrow$ Free energy of the system $F = -k_B T \log \Lambda_0(0)$
- Next largest eigenvalues $\rightarrow$ Correlation lengths

Problem: The QTM does not exist for continuum models!

- Continuum limit of the XXZ spin chain $\rightarrow$ Lieb-Liniger model (Kulish, Sklyanin 1979)
- Yang's thermodynamics from the XXZ spin chain QTM result (Seel, Bhattacharyya, Göhmann, Klümper 2007)

General strategy:

1. Find a suitable lattice embedding of the continuum model
2. Compute the thermodynamics of the lattice model using QTM
3. Take the continuum limit

The $q = 3$ Perk-Schultz spin chain

Hamiltonian

$$\mathcal{H}_{PS} = J\varepsilon_1 \sum_{j=1}^{L} \left(\cos\gamma \sum_{a=1}^{3} \varepsilon_a e_{aa}^{(j)} e_{aa}^{(j+1)} + \sum_{\substack{a,b=1\\ a\neq b}}^{3} e_{ab}^{(j)} e_{ba}^{(j+1)} + i \sin\gamma \sum_{\substack{a,b=1\\ a\neq b}}^{3} \mathrm{sign}(a-b) e_{aa}^{(j)} e_{bb}^{(j+1)} \right) - \sum_{j=1}^{L} \sum_{a=1}^{3} h_a e_{aa}^{(j)} ,$$

L is the number of lattice sites, $\gamma \in (0, \pi)$ determines the anisotropy, $J > 0$ coupling strength, h_a chemical potentials. $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the grading ($\varepsilon = \pm 1$).
Fundamental spin model of the $q = 3$ Perk-Schultz R-matrix

$$\mathrm{R}(v, w) = \sum_{a=1}^{3} \mathrm{R}_{aa}^{aa}(v, w) e_{aa} \otimes e_{aa} + \sum_{\substack{a,b=1\\ a\neq b}}^{3} \mathrm{R}_{ab}^{ab}(v, w)\, e_{aa} \otimes e_{bb} + \sum_{\substack{a,b=1\\ a\neq b}}^{3} \mathrm{R}_{ba}^{ab}(v, w)\, e_{ab} \otimes e_{ba} ,$$

$$\mathrm{R}_{aa}^{aa}(v, w) = \frac{\sin[\gamma + \varepsilon_a (v - w)]}{\sin\gamma} , \quad \mathrm{R}_{ab}^{ab}(v, w) \underset{a\neq b}{=} \frac{\sin(v - w)}{\sin\gamma} , \quad \mathrm{R}_{ba}^{ab}(v, w) \underset{a\neq b}{=} e^{i \mathrm{sgn}(a-b)(v-w)} ,$$

$(e_{ab})_{ij} = \delta_{ia}\delta_{jb}$ is the canonical basis in the space of 3-by-3 matrices.

- hyperbolic regime: Perk, Schultz (1981), Babelon, Vega, Viallet (1982),
- trigonometric regime: de Vega, Lopes (1991)

Continuum limit

Perk-Schultz spin chain	Gaudin-Yang model
lattice constant $\delta \to \mathcal{O}(\epsilon^2)$	particle mass $m = 1/2$
number of lattice sites $L \to \mathcal{O}(1/\epsilon^2)$	physical length $L_{GY} = L\delta$
interaction strength $J > 0$	repulsion strength $c = \epsilon^2/\delta$
chemical potential $h_1 \to \mathcal{O}(\epsilon^2)$	chemical potential $\mu = \frac{J\epsilon^2}{\delta^2} - \frac{h_1}{\delta^2} + \frac{J\epsilon^4}{4\delta^2} + \frac{1}{2\delta^2}(h_3 + h_2)$
chemical potentials $h_2, h_3 \to \mathcal{O}(\epsilon^4)$	magnetic field $H = (h_3 - h_2)/(2\delta^2)$
inverse temperature β	inverse temperature $\overline{\beta} = \beta\delta^2$
anisotropy $\gamma = \pi - \epsilon$	

Bosonic case: $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$=$(- - -)$ Fermionic case: $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$=$(- + +)$
Spectral parameter $\frac{\epsilon}{\delta}\lambda = k$;

- BAEs (Perk-Schultz spin chain) $\to$ BAE (Gaudin-Yang model)
- One-particle energy $\overline{\beta}\overline{e}_0(\lambda) \to \beta e_0(k)$
- $e^{\beta E_0} Z(h_1, h_2, h_3, \beta) \to \mathcal{Z}(\mu, H, \bar{\beta})$.

Low-T thermodynamics of the PS spin chain $\to$ Thermodynamics of the Gaudin-Yang model at all T

Thermodynamics: Bosonic system I

Grandcanonical potential (Klümper and O.I.P 2011,2015)

$$\phi(\mu, H, \beta) = -\frac{1}{2\pi\beta}\int_{\mathbb{R}} [\ln(1 + a_1(k)) + \ln(1 + a_2(k))] \, dk \,,$$

$a_{1,2}(k)$ auxiliary functions

$$\ln a_1(k) = -\beta(k^2 - \mu - H) + K_0 * \ln[1 + a_1](k) \qquad + K_2 * \ln[1 + a_2](k - i\varepsilon) \,,$$
$$\ln a_2(k) = -\beta(k^2 - \mu + H) + K_1 * \ln[1 + a_1](k + i\varepsilon) + K_0 * \ln[1 + a_2](k) \,,$$

with $\varepsilon \to 0$ and

$$K_0(k) = \frac{1}{2\pi}\frac{2c}{k^2 + c^2} \,, \quad K_1(k) = \frac{1}{2\pi}\frac{c}{k(k + ic)} \,, \quad K_2(k) = \frac{1}{2\pi}\frac{c}{k(k - ic)} \,.$$

Valid for all values of μ, H, T and c.

Thermodynamics: Bosonic system II

Known limiting cases

- Noninteracting limit: $c \to 0$
  $\lim_{c\to 0} K_2(k - i\varepsilon) = \lim_{c\to 0} K_1(k + i\varepsilon) = 0$ and $\lim_{c\to 0} K_0^B(k - k') = \delta(k - k')$ the NLIEs decouple

$$\ln a_1(k) = -\beta(k^2 - \mu - H) + \ln(1 + a_1(k)), \quad \ln a_2(k) = -\beta(k^2 - \mu + H) + \ln(1 + a_2(k)).$$

$$\phi(\mu, H, \beta) = \frac{1}{2\pi\beta} \int_{\mathbb{R}} \left[\ln(1 - e^{-\beta(k^2-\mu-H)}) + \ln(1 - e^{-\beta(k^2-\mu+H)})\right] dk$$

- Strong magnetic field (polarized system)
  For $H \to \infty$, $a_2(k) \sim 0$

$$\phi(\mu, H, \beta) = -\frac{1}{2\pi\beta} \int_{\mathbb{R}} \ln(1 + a_1(k))\, dk, \quad \log a_1(k) = -\beta(k^2 - \mu - H) + K_0 * \ln[1 + a_1](k).$$

  Yang-Yang thermodynamics of the Lieb-Liniger model. Same result is obtained for H fixed and $T \to 0$ proving the ferromagnetic nature of the groundstate.

- Impenetrable particles: $c \to \infty$
  Same result as for impenetrable fermions (Takahashi 1971). Checked numerically.

$$\phi_F(\mu, H, \beta) = -\frac{1}{2\pi\beta} \int_{\mathbb{R}} dk \ln\left(1 + 2\cosh(\beta H) e^{-\beta(k^2-\mu)}\right).$$

Thermodynamics: Fermionic system I

Grandcanonical potential (Klümper and O.I.P 2016)

$$\phi(\mu, H, \beta) = -\frac{1}{2\pi\beta}\int_{\mathbb{R}} [\ln(1 + a_1(k)) + \ln(1 + a_2(k))]\; dk\,,$$

$a_{1,2}(k)$ auxiliary functions

$$\ln a_1(k) = -\beta(k^2 - \mu - H) + K_2 * \ln[1 + a_2](k - i\varepsilon)\,,$$
$$\ln a_2(k) = -\beta(k^2 - \mu + H) + K_1 * \ln[1 + a_1](k + i\varepsilon)\,,$$

with $\varepsilon \to 0$ and

$$K_1(k) = \frac{1}{2\pi}\frac{c}{k(k + ic)}\,,\quad K_2(k) = \frac{1}{2\pi}\frac{c}{k(k - ic)}\,.$$

Thermodynamics: Fermionic system II

Known limiting cases

- Noninteracting limit: $c \to 0$
  $\lim_{c\to 0} K_2(k - i\varepsilon) = \lim_{c\to 0} K_1(k + i\varepsilon) = 0$ the NLIEs become

$$\ln a_1(k) = -\beta(k^2 - \mu - H)\,, \;\; \ln a_2(k) = -\beta(k^2 - \mu + H)\,.$$

$$\phi(\mu, H, \beta) = -\frac{1}{2\pi\beta}\int_{\mathbb{R}} \left[\ln(1 + e^{-\beta(k^2-\mu-H)}) + \ln(1 + e^{-\beta(k^2-\mu+H)})\right] dk$$

- Strong magnetic field (polarized system)
  For $H \to \infty$, $a_2(k) \sim 0$

$$\phi(\mu, H, \beta) = -\frac{1}{2\pi\beta}\int_{\mathbb{R}} \ln(1 + a_1(k))\, dk\,, \;\; \log a_1(k) = -\beta(k^2 - \mu - H).$$

  Free fermions.

- Impenetrable particles: $c \to \infty$ (Takahashi 1971)
  Checked numerically.

$$\phi_F(\mu, H, \beta) = -\frac{1}{2\pi\beta}\int_{\mathbb{R}} dk\, \ln\left(1 + 2\cosh(\beta H) e^{-\beta(k^2-\mu)}\right)\,.$$

Tan contact. I

Momentum distribution of models with zero range spin-independent interactions exhibit a universal

$$\lim_{k\to\infty} n_\sigma(k) \sim \frac{C}{k^4} \qquad \text{C is called contact}$$

For the fermionic GY model C is the same for $\sigma = \{\uparrow, \downarrow\}$ and is experimentally measurable

- Lieb-Liniger model (Olshanii, Dunjko, 2004)
- 3D spin 1/2 fermions (S. Tan, 2005)

For the fermionic Gaudin-Yang model (we can also include a trapping potential $V(x) = m\omega^2 x^2/2$) the contact is defined as ($c = -2/a_{1D}$ with a_{1D} the scattering length)

$$C = \frac{4}{a_{1D}^2} \int dx \, \langle \Psi_\uparrow^\dagger(x)\Psi_\downarrow^\dagger(x)\Psi_\downarrow(x)\Psi_\uparrow(x)\rangle = c^2 \int dx \, \langle \Psi_\uparrow^\dagger(x)\Psi_\downarrow^\dagger(x)\Psi_\downarrow(x)\Psi_\uparrow(x)\rangle \, ,$$

$\langle \, \rangle$ we denote the zero or finite temperature expectation value. The contact is related to the interaction energy $I = 2c \int dx \, \langle \Psi_\uparrow^\dagger(x)\Psi_\downarrow^\dagger(x)\Psi_\downarrow(x)\Psi_\uparrow(x)\rangle$ and the local correlation function of opposite spins

$$g_{\uparrow\downarrow}^{(2)}(x) = 4\frac{\langle \Psi_\uparrow^\dagger(x)\Psi_\downarrow^\dagger(x)\Psi_\downarrow(x)\Psi_\uparrow(x)\rangle}{n(x)^2} = 4\frac{\langle \Psi_\uparrow^\dagger(x)\Psi_\downarrow^\dagger(x)\Psi_\downarrow(x)\Psi_\uparrow(x)\rangle}{(n_\uparrow(x) + n_\downarrow(x))^2} \, .$$

Tan contact. II

Homogeneous case $V(x) = 0$. Contact and interaction energy per length

$$\mathcal{C} = \frac{c^2}{4} n^2 g^{(2)}_{\uparrow\downarrow}(0)\,, \quad \mathcal{I} = \frac{c}{2} n^2 g^{(2)}_{\uparrow\downarrow}(0)\,.$$

Using the Hellmann-Feynman theorem (ϕ is the grandcanonical potential per length)

$$g^{(2)}_{\uparrow\downarrow}(0) = \frac{2}{n^2}\frac{\partial\phi}{\partial c}$$

Tan relations (Tan 2005; Barth and Zwerger 2011):

- Tan adiabatic theorem (Hellman-Feynman theorem)

$$\frac{dE}{da_{1D}} = \left\langle \frac{\partial \mathcal{H}_{GY}}{\partial a_{1D}} \right\rangle = C$$

  variation of the total energy E with respect to the scattering length.
- Pressure relation: $p = 2\mathcal{E} - 2\mathcal{C}/c$
- Energy relation (inhomogeneous system): $E = 2\langle V \rangle + C/c$

Results at T=0. Dependence on the coupling strength I

At T=0 the groundstate is characterized by

$$\rho_c(k) = \frac{1}{2\pi} + \int_{-\lambda_0}^{\lambda_0} b_1(k-\lambda)\rho_s(\lambda)\, d\lambda$$

$$\rho_s(\lambda) = \int_{-k_0}^{k_0} b_1(\lambda - k)\rho_c(k)\, dk - \int_{-\lambda_0}^{\lambda_0} b_2(\lambda-\mu)\rho_s(\mu)\, d\mu$$

where $b_m(k) = mc/[2\pi[(mc)^2/4 + k^2]]$. k_0 and λ_0 are parameters which fix the values of the density of spin down particles and energy per length via

$$n = \int_{-k_0}^{k_0} \rho_c(k)\, dk\,, n_\downarrow = \int_{-\lambda_0}^{\lambda_0} \rho_s(\lambda)\, d\lambda\,, \mathcal{E} = \int_{-k_0}^{k_0} k^2\rho_c(k)\, dk\,.$$

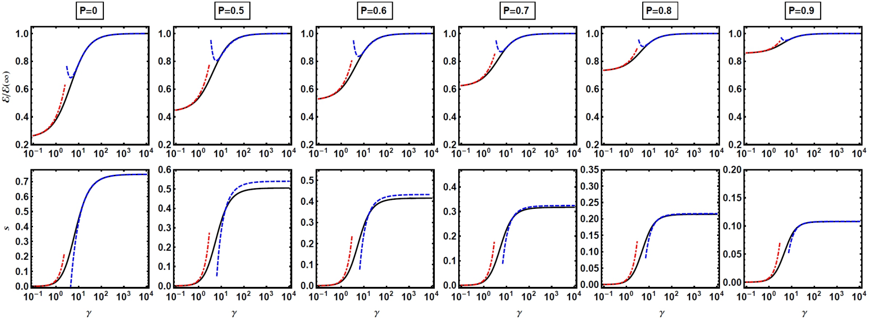
Results at T=0. Dependence on the coupling strength II

Asymptotic expansions (Guan and Ma, 2012)

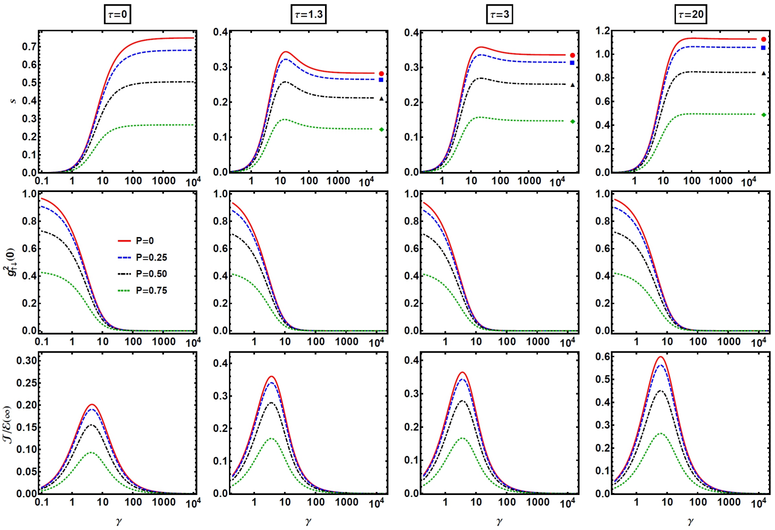
$$\mathcal{E}=\begin{cases}\frac{n^3\pi^2}{3}\left[1-\frac{4\ln 2}{\gamma}+\frac{12(\ln 2)^2}{\gamma^2}-\frac{32(\ln 2)^3}{\gamma^3}+\frac{\pi^2\zeta(3)}{\gamma^3}\right], & P=0\,,\\ \frac{n^3\pi^2}{3}\left[1-\frac{4(1-P)}{\gamma}+\frac{12(1-P)^2}{\gamma^2}-\frac{32(1-P)^3}{\gamma^3}+\frac{16\pi^2(1-P)}{5\gamma^3}\right], & P\geq 0.5\,,\end{cases}\qquad (\gamma\gg 1)$$

$$\mathcal{E}=\frac{n^3\pi^2(1-P)^3}{24}+\frac{n^3\pi^2(1+P)^3}{24}+\gamma\frac{n^3(1-P^2)}{2}\,,\qquad (\gamma\ll 1)$$

$$g^{(2)}_{\uparrow,\downarrow}(0)=\frac{2}{n^3}\frac{d\mathcal{E}}{d\gamma}\,,\quad \mathcal{C}=\frac{n}{2}\gamma^2\frac{d\mathcal{E}}{d\gamma}\,,\quad \mathcal{I}=\gamma\frac{d\mathcal{E}}{d\gamma}\qquad (\gamma=c/n)$$

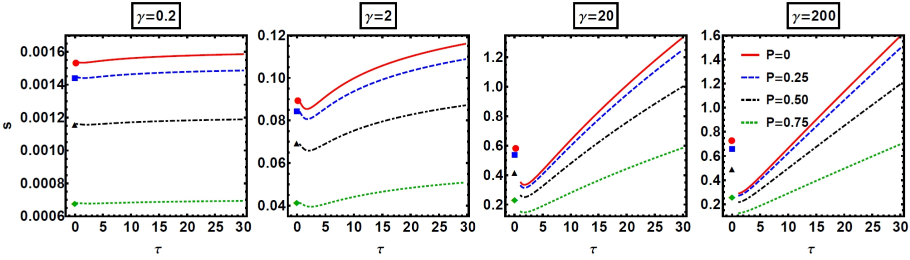


Results at T>0. Dependence on the coupling strength



- Al low T the contact presents a local maximum
- $g^{(2)}_{\uparrow,\downarrow}(0) \to 0$ like $1/\gamma^2$

Results at T>0. Dependence on temperature



- Crossover of the momentum distribution (Cheianov, Smith and Zvonarev 2005, $\gamma = \infty$) at strong coupling. As we increase the temperature the momentum distribution becomes **narrower**.
- There are two temperature scales $T_F = \pi^2 n^2$ and $T_0 = T_F/\gamma \rightarrow$ two different quantum regimes: $T < T_0$ Tomonaga-Luttinger liquid theory and $T_0 \ll T \ll T_F$ incoherent spin Luttinger theory (Cheianov and Zvonarev 2004, Fiete and Balents 2004).
- In the strong coupling limit and for $T_0 \ll T \ll T_F$ the charge degrees of freedom are effectively frozen while the spin degrees of freedom are strongly “disordered".
- At high-T the contact depends linearly on the reduced temperature

Bose-Fermi mixture [Mildly speculative]

Grandcanonical potential

$$\phi(\mu, H, \beta) = -\frac{1}{2\pi\beta}\int_{\mathbb{R}} [\ln(1+a_1(k)) + \ln(1+a_2(k))]\, dk\,,$$

$$\begin{pmatrix} \ln a_1 \\ \ln a_2 \end{pmatrix}(k) = \begin{pmatrix} -\beta(k^2-\mu-H) \\ -\beta(k^2-\mu+H) \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} * \begin{pmatrix} \ln[1+a_1] \\ \ln[1+a_2] \end{pmatrix}(k)$$

- Bosonic $\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} K_0 & K_2 \\ K_1 & K_0 \end{pmatrix}$
- Fermionic $\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} 0 & K_2 \\ K_1 & 0 \end{pmatrix}$

$$K_0(k) = \frac{1}{2\pi}\frac{2c}{k^2+c^2}\,,\quad K_1(k) = \frac{1}{2\pi}\frac{c}{k(k+ic)}\,,\quad K_2(k) = \frac{1}{2\pi}\frac{c}{k(k-ic)}\,.$$

- Bose-Fermi mixture $\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} K_0 & K_2 \\ K_1 & 0 \end{pmatrix}$

N-components system [Highly speculative]

Grandcanonical potential

$$\phi(\mu_1,\cdots,\mu_N,\beta) = -\frac{1}{2\pi\beta}\sum_{i=1}^{N}\int_{\mathbb{R}} \ln(1+a_i(k))\,dk\,,$$

$$\begin{pmatrix} \ln a_1 \\ \ln a_2 \\ \vdots \\ \ln a_N \end{pmatrix}(k) = \begin{pmatrix} -\beta(k^2-\mu_1) \\ -\beta(k^2-\mu_2) \\ \vdots \\ -\beta(k^2-\mu_N) \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{pmatrix} * \begin{pmatrix} \ln[1+a_1] \\ \ln[1+a_2] \\ \vdots \\ ln[1+a_N] \end{pmatrix}(k)$$

$N=4$

- Bosonic $\begin{pmatrix} K_0 & K_2 & K_2 & K_2 \\ K_1 & K_0 & K_2 & K_2 \\ K_1 & K_1 & K_0 & K_2 \\ K_1 & K_1 & K_1 & K_0 \end{pmatrix}$
- Fermionic $\begin{pmatrix} 0 & K_2 & K_2 & K_2 \\ K_1 & 0 & K_2 & K_2 \\ K_1 & K_1 & 0 & K_2 \\ K_1 & K_1 & K_1 & 0 \end{pmatrix}$

Conclusions and outlook

Summary

- New thermodynamic description of the bosonic and fermionic Gaudin-Yang model
- Nonmonotonic behaviour of the contact as a function of the coupling strength and temperature. Momentum reconstruction in the Tonks-Girardeau regime

Outlook

- Improved numerical approach
- 3-component case
- Correlation lengths

THANK YOU !