Minimum black hole mass from colliding Gaussian packets

R. Casadio[a,b*], O. Micu[c†] and A. Orlandi[a,b‡]

[a]*Dipartimento di Fisica, Università di Bologna*
via Irnerio 46, 40126 Bologna, Italy

[b]*Istituto Nazionale di Fisica Nucleare, Sezione di Bologna*
via Irnerio 46, 40126 Bologna, Italy

[c]*Institute of Space Science*
P.O. Box MG-23, Ro 077125 Bucharest-Magurele, Romania

May 30, 2012

Abstract

We study the formation of a black hole in the collision of two Gaussian packets. Rather than following their dynamical evolution in details, we assume a horizon forms when the mass function for the two packets becomes larger than half the flat areal radius, as it would occur in a spherically symmetric geometry. This simple approximation allows us to determine the existence of a minimum black hole mass solely related to the width of the packets. We then comment on the possible physical implications, both in classical and quantum physics, and models with extra spatial dimensions.

1 Introduction

The subject of gravitational collapse and black hole formation in classical general relativity dates back to the seminal papers of Oppenheimer and co-workers [1]. Since then, the literature has grown immensely, but the topic remains remarkably complex (for an overview, see, e.g., Refs. [2, 3], and references therein).

---

[*]casadio@bo.infn.it

[†]micu.octavian@gmail.com

[‡]alessio.j.orlandi@gmail.com

One thing we can safely claim is that gravity will come into play strongly whenever two localized matter states approach each other to a sufficiently short distance. Indeed, in 1972, K. Thorne proposed the so-called *hoop conjecture* [4]:

> A black hole forms whenever the impact parameter b of two colliding objects (of negligible spatial extension) is shorter than the radius of the would-be-horizon (roughly, the Schwarzschild radius, if angular momentum can be neglected) corresponding to the total energy M of the system, that is for [1]

$$b \lesssim \frac{2 \, \ell_\mathrm{p} \, M}{m_\mathrm{p}} \ . \tag{1.1}$$

The conjecture has been checked and verified in a variety of situations. Of course, it was initially formulated for black holes of (at least) astrophysical size [5], for which the very concept of a classical background metric and related horizon structure should be reasonably safe (for a review of some problems related to the hoop conjecture, see the bibliography in Ref. [6]).

Whether the hoop conjecture can also be trusted for masses approaching the Planck size, however, becomes more questionable. In fact, for such (relatively) small masses, quantum effects may not be neglected (for a recent discussion, see, e.g., Ref. [7]) and it is conceivable that (semi-)classical black holes must be replaced by some new kinds of object, generically referred to as "quantum black holes" (see, e.g., Refs. [8, 9]). Since we do not have any experimental insight for such a high-energy regime, it is conceptually difficult to conceive a theory for these objects, and it might be a good starting point just to push our established knowledge to the limit.

As we just recalled, it is of particular conceptual interest to study the possibility of black hole production in high-energy collisions [10, 11, 12]. Along these lines, Dvali and co-workers [13] recently went on to conjecture that the high-energy limit of all physically relevant quantum field theories involves the formation of a (semi)classical state (to wit, black hole formation for gravity), which should automatically suppress trans-Planckian quantum fluctuations. This idea extends the concept of a quantum uncertainty principle generalized to include gravity, as was considered, for example in Refs. [14, 15]. In this context, it can also be inferred that the mass of microscopic black holes must be quantized, and admit a minimum value [16] (for more general cases, see also Ref. [17]).

All of the above conjectures would be conspicuously substantiated if we could solve the extremely complex dynamics of colliding Standard Model particles, including the effect of the gravitational interaction, around the Planck scale [18]. Since the completion of this task appears still far, we shall here study a toy model, which assumes the validity of results known to hold in spherically symmetric (classical) systems, and reproduces the above formula (1.1) to a good approximation. Bearing on this consistency check, we

---

[1] We shall use units with $c = \hbar = 1$, and always display the Newton constant $G = \ell_\mathrm{p}/m_\mathrm{p}$, where ℓ_p and m_p are the Planck length and mass, respectively.

shall then be able to derive the existence of a minimum black hole mass, provided there exists a minimum (fundamental or testable) length [15, 19, 20, 21, 22]. Beside a purely conceptual interest, the existence of this mass threshold may have phenomenological implications in models with extra spatial dimensions [23, 24], where the fundamental (gravitational) length corresponds to energy scales potentially as low as a few TeV's.

2 Toy model for black hole production

Classical horizon formation is a relatively well-understood process in spherically symmetric space-times. We can write a general spherically symmetric metric $g_{\mu\nu}$ as

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + r^2(x^i)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\ , \tag{2.1}$$

where r is the areal coordinate and $x^i = (x^1, x^2)$ are coordinates on surfaces with θ and ϕ constant. The location of a trapping horizon is determined by the equation (see, e.g., [25] and references therein)

$$0 = g^{ij}\,\nabla_i r\,\nabla_j r = 1 - \frac{2\,m}{r}\ , \tag{2.2}$$

where $\nabla_i r$ is the covector perpendicular to surfaces of constant area $\mathcal{A} = 4\,\pi\,r^2$. The quantity $m = \ell_{\mathrm{p}}\,M/m_{\mathrm{p}}$ is the active gravitational (or Misner-Sharp) mass function representing the total energy enclosed within a sphere of radius r. For example, is we set $x^1 = t$ and $x^2 = r$, the function M is explicitly given by the integral of the matter density $\rho = \rho(x^i)$ weighted by the flat metric volume measure,

$$M(t,r) = \frac{4\,\pi}{3} \int_0^r \rho(t,\bar{r})\,\bar{r}^2\,\mathrm{d}\bar{r}\ , \tag{2.3}$$

as if the space inside the sphere were flat. Of course, it is in general very difficult to follow the dynamics of a given matter distribution and verify the existence of surfaces satisfying Eq. (2.2).

Let us consider two equal Gaussian packets of width ℓ moving toward each other in (asymptotically) flat space-time, say along the z direction of a cartesian system (see Fig. 1). We neglect the details of their spatial extension along z, and just consider the configuration of the two packets at the instant when their centers are at the minimum distance, say $2\,b$. We interpret the square of such packets as the "energy density" of two particles, including the kinetic contribution,

$$\begin{aligned}\rho_{\pm}(x,y) &= \frac{\rho_0}{\pi\,\ell^2}\,\exp\left\{-\frac{(x\pm b)^2 + y^2}{\ell^2}\right\}\\ &= \frac{\rho_0}{\pi\,\ell^2}\,\exp\left\{-\frac{r^2 \pm 2\,b\,r\,\cos(\theta) + b^2}{\ell^2}\right\} = \rho_{\pm}(r,\theta)\ ,\end{aligned} \tag{2.4}$$
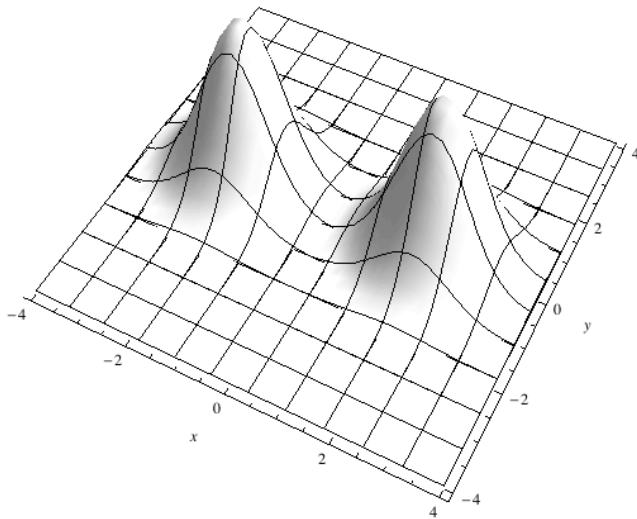
3

Figure 1: Gaussian packets with $b = 2\,\ell$ at their minimum distance. All lengths are in units of ℓ.

where we introduced cylindrical coordinates on the plane transverse to z, with origin at the midpoint between the packets, in $x = y = 0$. The Gaussians are normalized in the x-y plane, so that the positive ρ_0 parametererizes the total energy, possibly encoding the details of the distribution along the "neglected" direction z.

If the system were spherically symmetric around the chosen origin, the metric would reduce to the form (2.1). Its mass function (2.3) would then be given by

$$
\begin{aligned}
M(r) \;&\propto\; \int_0^r r'\,\mathrm{d}r' \int_{-\pi}^{+\pi} \mathrm{d}\theta \left[\rho_+(r',\theta) + \rho_-(r',\theta) \right] \\
&=\; 4\,\pi\,\rho_0 \int_0^r \frac{r'\,\mathrm{d}r'}{\pi\,\ell^2}\, e^{-\frac{r'^2 + b^2}{\ell^2}}\, I_0\!\left(\frac{2\,b\,r'}{\ell^2} \right) \\
&=\; \rho_0 \frac{\ell^2}{b^2} \int_0^{\frac{2\,b\,r}{\ell^2}} s\,\mathrm{d}s\, e^{-\frac{\ell^4\,s^2 + 4\,b^4}{4\,b^2\,\ell^2}}\, I_0(s) \;,
\end{aligned}
\qquad (2.5)
$$

where I_0 is the Bessel function of order 0, and ρ_0 only appears as a multiplicative factor with dimensions of a mass. Let us assume we can still apply this concept of mass function and proceed as if the system could be reasonably approximated by arguments only properly defined in spherical symmetry.
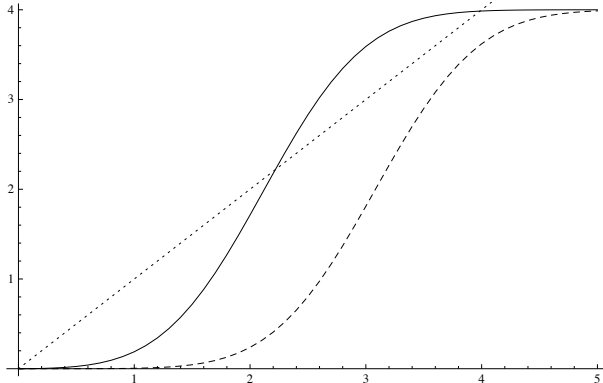
4

Figure 2: Horizon function $R = R(r)$ for $\rho_0 = \ell$ with $b = 2\,\ell$ (solid line) and $b = 3\,\ell$ (dashed line). Possible horizons appear where the curves intersect the dotted line $R = r$. All lengths are in units of ℓ.

The above integral can be performed numerically, for any given values of the parameters. In Fig. 2, we plot the resulting horizon function

$$R = \frac{2\,\ell_{\mathrm{p}}\,M(r)}{m_{\mathrm{p}}} \ , \tag{2.6}$$

for $\rho_0 = \ell$ with $b = 2\,\ell$ (solid line) and $b = 3\,\ell$ (dashed line). The former case shows the appearance of a horizon where $R(r) = r$, whereas the latter does not. In fact, the horizon function $R = R(r)$ either intersects the axis r twice or not at all. The inner intersection, say $r = R_-$, occurs when $R(r) < r$ grows faster than r, whereas the second one, $r = R_+$, appears for $R(r) > r$ approaching the asymptotic value. Since $R(r) > r$ for $R_- < r < R_+$, it is the intersection R_+ which might represent the outer horizon, and prevent signals originating from the system from reaching out (whereas R_- could be just a spurious artifact of our toy model and we thus do not discuss it any further). Note that our calculation falls somewhat short of the hoop conjecture: according to Eq. (1.1), $b = 3\,\ell < 4\,\ell$ should be close enough for black hole formation, but this discrepancy could be easily removed by suitably normalizing the Gaussian distributions so as to take into account the spread along the z direction more precisely.

Upon varying the parameters ℓ, b and ρ_0, one comes to two interesting results:

1. For impact factors $b \lesssim \ell$, the horizon function quickly approaches an asymptotic shape and is basically unaffected by the specific value of b (see Fig. 3). Moreover, for $b \lesssim \ell$ the collision is almost head-on and the appearance of a horizon seems to depend only on the ratio ρ_0/ℓ;

2. For fixed ℓ and b, there is a minimum value of ρ_0 (hence, a minimum value of M, say M_0) which shows the presence of a horizon (see Fig. 4).
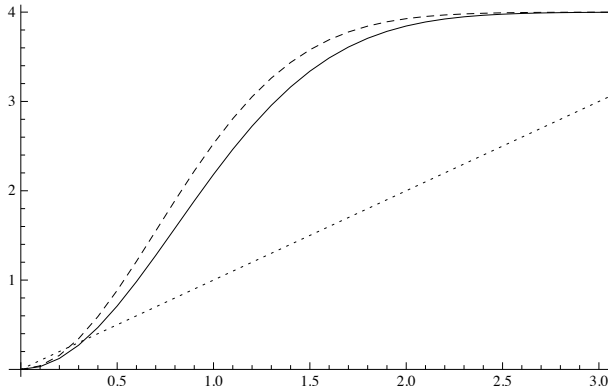
Figure 3: Horizon function $R = R(r)$ for $\rho_0 = \ell$ with $b = \ell/2$ (solid line) and $b = \ell/100$ (dashed line). All lengths are in units of ℓ.
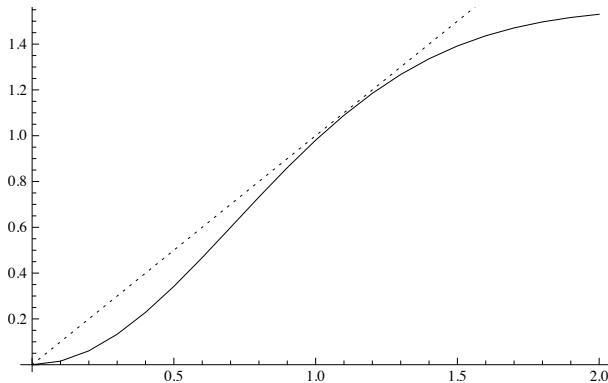

Figure 4: Horizon function $R = R(r)$ for $b = \ell/10$ and $\rho \simeq 2\,\ell/5$. All lengths are in units of ℓ.

Upon putting the two above results together, we can then draw the following conclusion: if there exists a minimum width ℓ for Gaussian densities, there is then a minimum black hole mass

$$M_0 \simeq m_{\mathrm{p}} \left(\frac{\ell}{\ell_{\mathrm{p}}} \right) \ . \tag{2.7}$$

Of course, it would make little sense to infer a precise numerical factor, given the simplicity of our calculation.

In particular, effects of the gravitational interaction between the two packets on their evolution have not been accounted for. This approximation might be still appropriate, if the packets do not significantly deviate from the Minkowskian trajectories and tidal effects remain small. Note, also, that we approximated the space-time geometry with a spherically symmetric metric, which necessarily implies the Schwarzschild geometry

6

sufficiently far away from the packets (say, for $r \gg \ell$). However, for a nonzero impact parameter b, the resulting black hole must have non-vanishing angular momentum (along the y axis). Moreover, the Gaussian packets may be meant to represent Standard Model particles (or partons). In our calculation, we did not assume any type of charges for the incoming waves and did not have to deal with their conservation laws. A better description would generally require using the more complex Kerr(-Newman) metric. Another point is that, if the energies of the two colliding packets are very different in the chosen (inertial or freely falling) frame, the resulting object will have a mass smaller than the sum of the energies of the two packets, part of the total energy being recovered as kinetic energy [26].

Let us conclude this section by remarking that we have not assumed any specific physical meaning for the width ℓ. The result (2.7) is therefore general enough and can be applied to various theories of space-time. It is also clear that it will induce a modification in the black hole production cross section, qualitatively similar to the low-energy suppression assumed in Ref. [27].

3 Discussion and physical implications

One can look at the simplified process of black hole production described in the previous section from very different perspectives. From a classical, general relativity point of view, one can think of the Gaussian functions as the (smoothed) density profile of spatially extended systems, such as galaxies or nebulae. Clearly, if an amount of matter (dust, stars or energy in general) is concentrated inside its horizon radius, a black hole will form and the rest of the system will be "scattered" according to the usual (but still very complex) general relativistic dynamics [5].

However, if one is willing to extend the above model into quantum scales, our view must change considerably. In this context, one can either consider the Gaussian packets as "probability distributions", as per point-particle quantum mechanics, or as describing the "effective extension" of elementary particles, as per quantum field theory. In both cases, we must then ask ourselves what happens to the "portion" of particles that remains outside of the horizon. Due to the indivisibility of such particles, there are two possibilities:

i) in the first case, one might think that "if particles partly collapse, then they must collapse entirely" (perhaps with a probability proportional to the portion of wave-function included within the horizon radius). This means that the two Gaussian packets totally contribute to the formation of the black hole, which will have a mass equal to the sum of the energies of the two colliding particles in the chosen centre-of-mass frame, and a black hole is all that is left at the end of the process;

ii) in the second case, we can assume that only the part of the energy effectively included within the horizon radius will be swallowed inside the black hole, whereas the remaining energy is somehow scattered away in the shape of new particles. As it

7

can be seen from Fig. 2, the horizon function is larger than r anywhere in the interval from R_- to R_+. It is thus in principle possible for a black hole with radius anywhere in this range to form as a result of the collision, provided the difference between the total center-of-mass energy of the incoming packets and black hole energy is scattered in the form of other particles (maybe carrying away the Standard Model charges of the incoming particles).

In both events, there is a mass threshold, but the spectrum of black hole masses is continuous. It has been suggested that black hole masses are instead quantized [16, 17]. If this turned out to be the case, the second option seems more likely, since a black hole would be created with quantized mass and respect energy conservation by scattering away the remaining energy.

In four space-time dimensions, it is plausible that $\ell \sim \ell_p$ [14] and $M_0 \sim m_p \simeq 10^{16}$ TeV. This means that the above considerations remain rather speculative. However, there are possible phenomenological implications in the context of models in which gravity can also propagate along extra spatial dimensions. If the gravitational scale m_p is within the reach of today's particle physics experiments, as suggested by these scenarios, (quantum) black holes may be produced (and detected) in colliders and by cosmic rays impinging on the atmosphere. Our simple calculation then adds to the current picture by describing the collisions between elementary particles, here viewed as objects extended in our three-dimensional space, in the shape of Gaussian energy distributions. If details of the extra-dimensional metric can be assumed to leave the picture (qualitatively [2]) unaffected, we can again say that, if the impact parameter is small enough, a black hole will form, confirming once more the hoop conjecture. Moreover, what we stated above simply applies with a Planck scale lowered to an energy potentially within a few TeV's. In particular, it is possible that the entire initial energy will be transformed into the black hole mass, but the resulting black hole can also have a mass below this value. In the latter case, the energy difference would be radiated in the form of Standard Model particles or bulk gravitons. The phenomenological implications for black holes which are created and decay instantaneously are then not clear. If, however, black holes live (relatively) long [28, 29], this result implies two possible signatures:

a) on the one hand, when black holes are formed with masses near the maximum possible value (their horizon radius being approximately equal to R_+), the entire available energy falls into the black hole and no other particles are radiated. In this case the resulting black holes must carry all the Standard Model charges of the incoming particles. Their signatures in particle accelerators would thus be large quantities of missing energy and transverse momentum, associated with the simultaneous detection of a very heavy and charged particle. This would be a very distinctive event, if we assume that the incoming Gaussian packets represent two quarks, since the resulting electric charge

---

[2]Of course, precise numerical coefficients will vary depending on the bulk metric, but we shall leave a more detailed discussion of this issues for future investigations.

will have a fractional value [30];

b) on the other hand, if the black holes were formed with masses bellow the maximum value, the charge could be radiated in the form of Standard Model particles, and the only sign of the black hole existence would just be large amounts of missing energy and transverse momentum.

Let us conclude by remarking again that the angular momentum and charges of the colliding particles are totally missing in the final black hole state we used for our derivation. It is possible that the angular momentum is emitted during the collision, like the initial Standard Model charges. However, if the black hole retains (part of) it, it will be very important to determine its effects on the mass threshold (2.7), along with those induced by any gauge charges. These points are left for future investigations.

Acknowledgements

References

[1] J.R. Oppenheimer and H. Snyder, Phys. Rev. **56** (1939) 455; J.R. Oppenheimer and G.M. Volkoff, Phys. Rev. **55** (1939) 374.

[2] P.S. Joshi, "Gravitational Collapse and Spacetime Singularities," Cambridge Monographs on Mathematical Physics (Cambridge, 2007).

[3] J.D. Bekenstein, "Black holes: Physics and astrophysics. Stellar-mass, supermassive and primordial black holes," astro-ph/0407560.

[4] K.S. Thorne, "Nonspherical Gravitational Collapse: A Short Review," in *J.R. Klauder, Magic Without Magic,* San Francisco (1972), 231.

[5] P.D. D'Eath and P.N. Payne, Phys. Rev. D **46** (1992) 658; Phys. Rev. D **46** (1992) 675; Phys. Rev. D **46** (1992) 694.

[6] J.M.M. Senovilla, Europhys. Lett. **81** (2008) 20004.

[7] G.L. Alberghi, R. Casadio, O. Micu and A. Orlandi, JHEP **1109** (2011) 023.

[8] S.D.H. Hsu, Phys. Lett. B **555** (2003) 92,

[9] X. Calmet, D. Fragkakis and N. Gausmann, Eur. Phys. J. C **71** (2011) 1781; X. Calmet, W. Gong and S.D.H. Hsu, Phys. Lett. B **668** (2008) 20.

[10] D.M. Eardley and S.B. Giddings, Phys. Rev. D **66** (2002) 044011.

[11] P. Kanti, Lect. Notes Phys. **769** (2009) 387.

[12] S.B. Giddings and S.D. Thomas, Phys. Rev. D **65** (2002) 056010.

[13] G. Dvali, C. Gomez and A. Kehagias, JHEP **1111** (2011) 070; G. Dvali, G.F. Giudice, C. Gomez and A. Kehagias, JHEP **1108** (2011) 108.

[14] M. Maggiore, Phys. Lett. B **319** (1993) 83; A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. D **52** (1995) 1108; F. Scardigli, Phys. Lett. B **452** (1999) 39; F. Scardigli and R. Casadio, Class. Quant. Grav. **20** (2003) 3915; Int. J. Mod. Phys. D **18** (2009) 319.

[15] M. Bleicher, P. Nicolini, M. Sprenger, Eur. J. Phys. 33 (2012) 853.

[16] G. Dvali, C. Gomez and S. Mukhanov, "Black Hole Masses are Quantized," arXiv:1106.5894 [hep-ph].

[17] M. Visser, "Quantization of area for event and Cauchy horizons of the Kerr-Newman black hole," arXiv:1204.3138 [gr-qc].

[18] D. Amati, M. Ciafaloni and G. Veneziano, JHEP **0802** (2008) 049.

[19] P. Nicolini, Int. J. Mod. Phys. A **24** (2009) 1229.

[20] L.J. Garay, Int. J. Mod. Phys. A **10** (1995) 145.

[21] C.A. Mead, Phys. Rev. **135** (1964) B849.

[22] X. Calmet, M. Graesser and S.D.H. Hsu, Phys. Rev. Lett. **93** (2004) 211101; Int. J. Mod. Phys. D **14** (2005) 2195.

[23] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 429**, 263 (1998); Phys. Rev. D **59**, 0806004 (1999); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 436**, 257 (1998).

[24] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999); Phys. Rev. Lett. **83**, 3370 (1999).

[25] S.A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini and S. Zerbini, Class. Quant. Grav. **26** (2009) 062001.

[26] X. Calmet, L.I. Caramete and O. Micu, "Quantum Black Holes from Cosmic Rays," arXiv:1204.2520 [hep-ph].

[27] J. Mureika, P. Nicolini and E. Spallucci, Phys. Rev. D **85** (2012) 106007.

[28] R. Casadio and B. Harms, Int. J. Mod. Phys. A **17** (2002) 4635; R. Casadio, S. Fabi, B. Harms and O. Micu, JHEP **1002**, 079 (2010); R. Casadio, B. Harms and O. Micu, Phys. Rev. D **82**, 044026 (2010); R. Casadio and O. Micu, Phys. Rev. D **81**, 104024 (2010).

[29] L. Bellagamba, R. Casadio, R. Di Sipio and V. Viventi, Eur. Phys. J. C **72** (2012) 1957.

[30] J. Pinfold [MOEDAL Collaboration], CERN Cour. **50N4** (2010) 19.