DO-TH 11/29

Lepton number violating effects in neutrino oscillations

Sebastian Hollenberg
Fakultät für Physik, Technische Universität Dortmund,
D-44221 Dortmund, Germany

Octavian Micu
Institute for Space Sciences, P.O. Box MG23,
Ro 077125 Bucharest-Magurele, Romania

Palash B. Pal
Saha Institute of Nuclear Physics, Kolkata, India

Abstract

We develop a *non-adiabatic* perturbation theory for oscillations involving an arbitrary number of neutrino and antineutrino species, including the possibility of lepton number violation which we treat as a small effect. We interpret the physics of such an approach for the one generation case by introducing the notion of adiabaticity for neutrino and antineutrino oscillations in analogy to flavor oscillations. We find that in a CP-odd matter environment a small lepton number violation in vacuo can be enhanced. Eventually, we apply the perturbation theory to the two generation case and work out an example for manifestations of lepton number violation, which can be solved both perturbatively as well as analytically thereby further clarifying the nature of the perturbation expansion.

Over the last decade or so the physics of neutrino flavor oscillations has seen considerable developments. Experiments have found that the three neutrinos discovered so far undergo flavor oscillations. Two of the three associated flavor mixing angles have been measured, the third one has been constrained to be small, but potentially non-zero. Moreover, the two mass splittings between the neutrino mass eigenstates have been determined. Despite all progress, a few neutrino properties remain elusive. It is, for instance, yet unknown whether there is any CP violation in the lepton sector or whether neutrino mass eigenstates obey a normal or inverted hierarchy [1].

There are, however, also some experimental findings in neutrino oscillation experiments which cannot be reconciled with the standard description of neutrino flavor oscillations. The latter findings are therefore often referred to as *neutrino oscillation anomalies*. Such findings as reported by the LSND [2] and MiniBooNE [3, 4] collaborations have given rise to a plethora of theoretical speculations how to explain them. The latter range from additional sterile neutrino species, non-standard neutrino interactions [5] and extra spatial dimensions [6] to CPT violation in the lepton sector [7].

Neutrino oscillations depend on neutrino masses. One of the most important questions connected with neutrino masses is the possibility of lepton number violation and consequent Majorana nature of the neutrinos. Lepton number violation should therefore affect neutrino oscillations, and in fact the effects can come in two forms. First of all, in addition to flavor oscillations, one can have lepton number violating oscillations, i.e., processes which can be termed neutrino-antineutrino oscillations [8]. Second, the usual flavor oscillation probabilities might be modified by the presence of lepton number violating terms in the Lagrangian.

Despite the fact that the reported neutrino oscillation anomalies might not last and may eventually be refuted by future neutrino oscillation experiments, the question of whether physics beyond the established picture of neutrino flavor oscillations with similar signatures exists is ultimately an experimental one. In this letter we propose a generic parameterization for lepton number violating neutrino flavor oscillations in order to qualitatively understand their ramifications and identify deviations from the signals predicted by lepton number conserving oscillations.

Let us, for such purposes, start by writing down the most general form of the Schrödinger equation involving active neutrino and antineutrino fields in flavor space. We denote the left-chiral, active neutrino fields collectively by the boldfaced symbol $\boldsymbol{\nu} = (\nu_e, \nu_\mu, \nu_\tau, \ldots)$. The active antineutrino fields are right-chiral, and they are denoted by $\bar{\boldsymbol{\nu}} = (\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau, \ldots)$. The Schrödinger equation has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \boldsymbol{\nu}(t) \\ \bar{\boldsymbol{\nu}}(t) \end{pmatrix} = -i\mathbb{H}(t)\begin{pmatrix} \boldsymbol{\nu}(t) \\ \bar{\boldsymbol{\nu}}(t) \end{pmatrix}, \tag{1}$$

where $\mathbb{H}$ is the Hamiltonian. We assume that the energy of the neutrino beams are such that the sterile states, which are expected to be heavy if a see-saw mechanism is assumed to be responsible for neutrino mass generation, are not excited and thus dynamically decouple from neutrino flavor oscillations.

In the block form that has been used to write this equation, we can write the total Hamiltonian in the form

$$\mathbb{H}(t) = \begin{pmatrix} H(t) & 0 \\ 0 & \hat{H}(t) \end{pmatrix} + \begin{pmatrix} 0 & B(t) \\ B^\dagger(t) & 0 \end{pmatrix} \equiv \mathbb{H}_0(t) + \delta\mathbb{H}(t), \tag{2}$$

where we have separated out the block-diagonal and off-diagonal parts. Since the fields in $\boldsymbol{\nu}$ are assigned lepton number equal to $+1$, whereas those in $\bar{\boldsymbol{\nu}}$ have the opposite lepton number, it implies that $\mathbb{H}_0$ is the lepton number conserving part of the Hamiltonian, whereas $\delta\mathbb{H}$ is lepton number violating. While writing the explicit form for $\delta\mathbb{H}$, we have used the assumption that all neutrinos are stable particles, so that the Hamiltonian is Hermitian. This assumption also implies that the block matrices H and $\hat{H}$ are both Hermitian. Note, moreover, that H is generically different from $\hat{H}$, since neutrino and antineutrino fields can be subject to CP non-conserving interactions as is the case, for instance, in elastic forward scattering of neutrinos off background fields [9].

If CPT is a symmetry of the Hamiltonian, there are additional constraints on the elements of $\mathbb{H}$. These constraints can be identified by recognizing that the active antineutrino states are CPT transforms of the active neutrino states. In general, if an operator $\mathscr{O}(t)$ commutes with CPT denoted by the operator Θ, we can write

$$\langle \Theta a | \, \mathscr{O}(t) \, | \Theta b \rangle = \langle b | \, \mathscr{O}^\dagger(t) \, | a \rangle = \langle a | \, \mathscr{O}(t) \, | b \rangle^* \tag{3}$$

for arbitrary state vectors $|a\rangle$ and $|b\rangle$, and denoting the state $\Theta\,|b\rangle$ by $|\Theta b\rangle$. For the active neutral fermions which appear in Eq. (1), we can choose the ordering as well as the phases of the states in such a way that

$$\Theta\,|\nu_a\rangle = |\bar{\nu}_a\rangle \,. \tag{4}$$

Therefore, Eq. (3), along with the Hermiticity of the Hamiltonian, would then imply the relation

$$\langle \nu_a | \, \mathbb{H}(t) \, | \nu_b \rangle = \langle \Theta\nu_b | \, \mathbb{H}(t) \, | \Theta\nu_a \rangle = \langle \bar{\nu}_b | \, \mathbb{H}(t) \, | \bar{\nu}_a \rangle \,. \tag{5}$$

In terms of the block-diagonal notation introduced in Eq. (2), this can be written as

$$\hat{H}(t) = H^\top(t)\,. \tag{6}$$

Going through similar steps, we can also show that

$$B(t) = B^\top(t)\,, \tag{7}$$

i.e., the block B is symmetric. Of course lepton number violating neutrino-antineutrino oscillations can occur even in scenarios with CPT violation, in which case Eqs. (6) and (7) need not hold; but also for the case in which Eqs. (6) and (7) hold, i.e., CPT is an exact symmetry, still lepton number violating neutrino oscillations are possible.

For the purposes that concern us in this paper, it is more convenient to introduce the time evolution operator $\mathbb{U}(t,t_0)$ via

$$\begin{pmatrix} \boldsymbol{\nu}(t) \\ \bar{\boldsymbol{\nu}}(t) \end{pmatrix} = \mathbb{U}(t,t_0) \begin{pmatrix} \boldsymbol{\nu}(t_0) \\ \bar{\boldsymbol{\nu}}(t_0) \end{pmatrix}\,. \tag{8}$$

The time evolution equation thus takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{U}(t,t_0) = -i\mathbb{H}(t)\mathbb{U}(t,t_0). \tag{9}$$

Henceforth, we will often take $t_0 = 0$ and will omit it in the notation. Let us have a look at the CPT properties of the time evolution operator. CPT symmetry would imply that the evolution operator satisfies the condition

$$\Theta\mathbb{U}(t) = \mathbb{U}(-t)\Theta\,, \tag{10}$$

and therefore the analog of Eq. (3) would be

$$\langle\Theta a|\,\mathbb{U}(t)\,|\Theta b\rangle = \langle a|\,\mathbb{U}(-t)\,|b\rangle^*\,. \tag{11}$$

In particular, it says that

$$\langle\bar{\nu}_b|\bar{\nu}_a(t)\rangle = \langle\nu_b|\nu_a(-t)\rangle^* = \langle\nu_a|\nu_b(t)\rangle\;, \tag{12}$$

implying that the oscillation probability of an $\bar{\nu}_a$ going to $\bar{\nu}_b$ is the same as that of a ν_b going to ν_a:

$$P(\bar{\nu}_a \to \bar{\nu}_b; t) = P(\nu_b \to \nu_a; t)\,. \tag{13}$$

In addition, Eq. (11) also implies that

$$P(\nu_a \to \bar{\nu}_b; t) = P(\nu_b \to \bar{\nu}_a; t)\,. \tag{14}$$

In our analysis, we adopt the paradigm that lepton number violating neutrino-antineutrino couplings B are small compared to lepton number conserving terms H, $\hat{H}$. We shall, in fact, adopt B as a *small* quantity for a perturbative solution [10] for the time evolution. To this end, we decouple the evolution of neutrinos and antineutrinos from the evolution of the neutrino-antineutrino system by introducing a (subsidiary) time evolution operator $\mathbb{G}(t)$ for the neutrino and antineutrino systems. The latter is then used to transform the time evolution equation for $\mathbb{U}(t,t_0)$ to what we shall henceforth refer to as the *interaction picture*. Establishing this line of action, we write

$$\mathbb{U}(t,t_0) = \mathbb{G}(t,t_0)\,\mathbb{U}_\mathrm{I}(t,t_0)\,, \tag{15}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{G}(t,t_0) = -i\mathbb{H}_0(t)\mathbb{G}(t,t_0), \tag{16}$$

where the subscript I indicates the associated quantity in the interaction picture. It is readily seen that the time evolution equation in the interaction picture is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{U}_{\mathrm{I}}(t,t_0) = -i\mathbb{H}_{\mathrm{I}}(t)\mathbb{U}_{\mathrm{I}}(t,t_0)\,, \tag{17}$$

with

$$\mathbb{H}_{\mathrm{I}}(t) = \mathbb{G}^{-1}(t,t_0)\delta\mathbb{H}(t)\mathbb{G}(t,t_0). \tag{18}$$

Since the Hamiltonian $\mathbb{H}_0$ is lepton number conserving, the solution of Eq. (16) is obviously of the form

$$\mathbb{G}(t,t_0) \;=\; \begin{pmatrix} G(t,t_0) & 0 \\ 0 & \hat{G}(t,t_0) \end{pmatrix}. \tag{19}$$

Putting this solution into Eq. (18), we obtain

$$\mathbb{H}_{\mathrm{I}}(t) \;=\; \begin{pmatrix} 0 & G^{-1}(t,t_0)B(t)\hat{G}(t,t_0) \\ \hat{G}^{-1}(t,t_0)B^{\dagger}(t)G(t,t_0) & 0 \end{pmatrix}. \tag{20}$$

We can now perturbatively solve the time evolution equation in the interaction picture by means of the Magnus expansion [11]. We write its solution as a matrix exponential

$$\mathbb{U}_{\mathrm{I}}(t,t_0) = e^{\Omega(t,t_0)}\,, \tag{21}$$

where the Magnus operator $\Omega(t,t_0)$ is the sum of the so-called Magnus approximants $\Omega_i(t,t_0)$ according to

$$\Omega(t,t_0) \equiv \Omega_1(t,t_0) + \Omega_2(t,t_0) + \ldots\,, \tag{22}$$

which are *small* in an appropriate sense as we shall see in due course. Various methods have been worked out to calculate the Magnus approximants [12]. They are found to obey

$$\Omega_1(t,t_0) \;=\; -i\int_{t_0}^{t}\mathrm{d}\tau\;\mathbb{H}_{\mathrm{I}}(\tau), \tag{23}$$

$$\Omega_2(t,t_0) \;=\; -\frac{1}{2}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\;\Big[\mathbb{H}_{\mathrm{I}}(t_1),\mathbb{H}_{\mathrm{I}}(t_2)\Big], \tag{24}$$

up to second order in the perturbation. We shall see in our analysis that neutrino-antineutrino oscillation effects only enter the oscillation probabilities in second order perturbation theory.

Our aim is to comment on how lepton number violating effects manifest themselves. We assume that the lepton number conserving effects, encoded in $\mathbb{G}$, can be obtained either exactly or perturbatively by solving the underlying Schrödinger equation. The solution for the time evolution operator $\mathbb{U}(t,t_0)$ can then be written as

$$\mathbb{U}(t,t_0) = \mathbb{G}(t,t_0)e^{\Omega(t,t_0)}\,, \tag{25}$$

according to Eqs. (15) and (21). Inserting the block form of the interaction picture Hamiltonian from Eq. (20) into Eqs. (23) and (24), we find that the Magnus operator up to second order perturbation theory can be written as

$$\Omega^{(2)}(t,t_0) = \begin{pmatrix} \widetilde{C}(t,t_0) & -i\widetilde{B}(t,t_0) \\ -i\widetilde{B}^{\dagger}(t,t_0) & \widetilde{D}(t,t_0) \end{pmatrix}, \tag{26}$$

where we have introduced the shorthand notations

$$\widetilde{B}(t,t_0) \;\equiv\; \int_{t_0}^{t}\mathrm{d}\tau\; G^{-1}(\tau,\tau_0)B(\tau)\hat{G}(\tau,\tau_0), \tag{27}$$

$$\widetilde{C}(t,t_0) \;\equiv\; -\frac{1}{2}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\;\left\{\dot{\widetilde{B}}(t_1)\dot{\widetilde{B}}^{\dagger}(t_2) - \left(\dot{\widetilde{B}}(t_1)\dot{\widetilde{B}}^{\dagger}(t_2)\right)^{\dagger}\right\}, \tag{28}$$

$$\widetilde{D}(t,t_0) \;\equiv\; -\frac{1}{2}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\;\left\{\dot{\widetilde{B}}^{\dagger}(t_1)\dot{\widetilde{B}}(t_2) - \left(\dot{\widetilde{B}}^{\dagger}(t_1)\dot{\widetilde{B}}(t_2)\right)^{\dagger}\right\}. \tag{29}$$

The dot indicates a derivative with respect to time. Note that the quantities $\widetilde{C}(t,t_0)$ and $\widetilde{D}(t,t_0)$ are quadratic in $\widetilde{B}(t,t_0)$, i.e., quadratic in the lepton number violating parameter B in the Hamiltonian. Then, up to second order terms in B, we obtain

$$e^{\Omega(t,t_0)} \simeq 1+\Omega+\frac{1}{2}\Omega^2+\mathcal{O}(\Omega^3) \tag{30}$$

$$= \begin{pmatrix} 1+\mathcal{O}^{(2)}(t,t_0) & -i\widetilde{B}(t,t_0) \\ -i\widetilde{B}^\dagger(t,t_0) & 1+\hat{\mathcal{O}}^{(2)}(t,t_0) \end{pmatrix} + \cdots\,, \tag{31}$$

where

$$\mathcal{O}^{(2)}(t,t_0) = \widetilde{C}(t,t_0)-\frac{1}{2}\widetilde{B}(t,t_0)\widetilde{B}^\dagger(t,t_0)\,, \tag{32}$$

$$\hat{\mathcal{O}}^{(2)}(t,t_0) = \widetilde{D}(t,t_0)-\frac{1}{2}\widetilde{B}^\dagger(t,t_0)\widetilde{B}(t,t_0)\,. \tag{33}$$

These quantities show how lepton number violating neutrino-antineutrino mixing encoded in $B(t)$ affects the neutrino and antineutrino sectors respectively.

With these ingredients, we can now write down the oscillation probabilities of various kinds. Let us denote the oscillation probabilities in absence of lepton number violating terms by the notations

$$P_0(\nu_a\to\nu_b;t) = \left|G_{ba}(t)\right|^2,$$
$$P_0(\bar\nu_a\to\bar\nu_b;t) = \left|\hat{G}_{ba}(t)\right|^2. \tag{34}$$

Up to second order in lepton number violating terms, the probabilities would then read

$$P(\nu_a\to\nu_b;t) = P_0(\nu_a\to\nu_b;t)+2\mathrm{Re}\left\{G^*_{ba}(t)\times\left(G(t)\mathcal{O}^{(2)}(t)\right)_{ba}\right\}, \tag{35}$$

$$P(\bar\nu_a\to\bar\nu_b;t) = P_0(\bar\nu_a\to\bar\nu_b;t)+2\mathrm{Re}\left\{\hat{G}^*_{ba}(t)\times\left(\hat{G}(t)\hat{\mathcal{O}}^{(2)}(t)\right)_{ba}\right\}. \tag{36}$$

Note that there is no summation over recurring indices implied here or elsewhere in the paper. Thus the lepton number preserving oscillation probabilities get modified. On the other hand, there can also be lepton number violating oscillations now, and the probabilities of such oscillations are given by

$$P(\nu_a\to\bar\nu_b;t) = \left|\left(\hat{G}(t)\widetilde{B}^\dagger(t)\right)_{ba}\right|^2, \tag{37}$$

$$P(\bar\nu_a\to\nu_b;t) = \left|\left(G(t)\widetilde{B}(t)\right)_{ba}\right|^2. \tag{38}$$

These oscillation probabilities for lepton number preserving as well as violating neutrino (antineutrino) oscillations Eqs. (35–38) present the main result of our analysis. A few comments are in order.

First, we want to emphasize that this method for obtaining the oscillation probabilities does not involve the effective mixing matrix at any stage of the analysis. It is of course possible, in principle at least, to diagonalize the entire Hamiltonian including the lepton number violating terms and derive oscillation probabilities from the resulting eigenvalues and eigenstates. Our aim was to demonstrate how some of the essential features of such a solution can be obtained without performing such a diagonalization. Note also that this approach reduces a $2N\times 2N$ problem (N being the number of neutrino species involved), the diagonalization of the Hamiltonian, to a $N\times N$ problem, since the perturbation expansion only involves block entries from the original Hamiltonian $\mathbb{H}(t)$.

Truncating the exponential of the Magnus operator results in loss of unitarity. Since we truncate the series in a way that we only keep terms up to second order in the small perturbation $B(t)$, we also expect that unitarity of the time evolution operator is only conserved up to second order in this quantity. It is straightforward to verify this by invoking the unitarity condition on $\mathbb{U}(t,t_0)$. Note that such considerations also pertain to solutions for the neutrino and antineutrino evolution operators $G(t,t_0)$ and $\hat{G}(t,t_0)$, if their solution is obtained in a similar way, i.e., by truncating the exponential of the associated Magnus operator in a perturbative manner. The fact that we seek a perturbative expansion of the solution to the time evolution operator readily implies that the structure of the oscillation probabilities is of the same kind. However, it should be realized that the probabilities given here are viable approximations for times that satisfy $|B|t \ll 1$, where $|B|$ indicates the magnitude of any non-zero eigenvalue of B. Hence the loss of unitarity is small as it is triggered through the small quantity $\widetilde{B}(t)$, and will appear only at orders B^3 or higher which have been neglected in writing down the expression for various oscillation probabilities.

In fact, unitarity can be tested explicitly from the expressions of the oscillation probabilities given above. Take, for example, Eq. (35). It shows that

$$\begin{aligned}\sum_b P(\nu_a \to \nu_b;t) &= 1 + 2\mathrm{Re}\sum_b \left\{ G^*_{ba}(t) \times \Big(G(t)\mathcal{O}^{(2)}(t)\Big)_{ba} \right\} \\ &= 1 + 2\mathrm{Re}\Big(G^\dagger(t)G(t)\mathcal{O}^{(2)}(t)\Big)_{aa} = 1 + 2\mathrm{Re}\Big(\mathcal{O}^{(2)}(t)\Big)_{aa}\,, \end{aligned} \tag{39}$$

using the unitarity of the matrix G. Further, looking at the expression of $\mathcal{O}^{(2)}(t)$ in Eq. (32), we see that the diagonal elements of this matrix must have negative real part, since the diagonal elements of $\widetilde{C}$ are purely imaginary and those of $\widetilde{B}\widetilde{B}^\dagger$ must be real and positive. Thus, starting from a particular flavor of neutrino ν_a, the total probability for lepton number conserving oscillations is

$$\sum_b P(\nu_a \to \nu_b;t) = 1 - \Big(\widetilde{B}(t)\widetilde{B}^\dagger(t)\Big)_{aa}\,, \tag{40}$$

which is less than unity, as one should expect. It should also be noted, from Eq. (37), that with the same initial state, the total probability for lepton number violating oscillations is given by

$$\begin{aligned}\sum_b P(\nu_a \to \bar{\nu}_b;t) &= \sum_b \Big(\hat{G}(t)\widetilde{B}^\dagger(t)\Big)^*_{ba}\Big(\hat{G}(t)\widetilde{B}^\dagger(t)\Big)_{ba} = \Big(\widetilde{B}(t)\hat{G}^\dagger(t)\hat{G}(t)\widetilde{B}^\dagger(t)\Big)_{aa} \\ &= \Big(\widetilde{B}(t)\widetilde{B}^\dagger(t)\Big)_{aa}\,, \end{aligned} \tag{41}$$

using the unitarity of the matrix $\hat{G}$. Therefore

$$\sum_b P(\nu_a \to \nu_b;t) + \sum_b P(\nu_a \to \bar{\nu}_b;t) = 1\,, \tag{42}$$

as expected, implying that in presence of lepton number violation, the lepton number conserving oscillation probabilities should decrease, making room for lepton number violating oscillations.

The form of the oscillation probabilities given in Eqs. (35–38) is illustrative in the sense that for the limiting case of vanishing neutrino-antineutrino coupling $B(t) \to 0$ it readily reduces to the standard, lepton number conserving neutrino oscillation results. The appearance of the oscillation probabilities for $\nu_a \to \bar{\nu}_b$ and $\bar{\nu}_a \to \nu_b$ can also be interpreted in an intuitive way: reading Eqs. (37 – 38) from right to left they state that $\widetilde{B}^\dagger(t)$ ($\widetilde{B}(t)$) switches the initial neutrino (antineutrino) state to the associated antiparticle and $\hat{G}(t)$ ($G(t)$) then evolves the antiparticle (particle) state until its detection. So neutrino-antineutrino oscillations are clearly a signal for lepton number violation in the neutrino sector. Note that this statement is a general one: all oscillation probabilities come with $\widetilde{B}(t)$ and derivatives thereof; put another way, in an approach in which neutrino-antineutrino couplings are treated as a small perturbation to all other potential enhancements in the neutrino and antineutrino sector respectively, one cannot have modifications in the neutrino-neutrino and antineutrino-antineutrino probabilities without introducing neutrino-antineutrino conversions at the same time. The oscillation probabilities for $\nu_a \to \bar{\nu}_b$ and $\bar{\nu}_a \to \nu_b$ are generically different by virtue of the generic difference between $\hat{G}(t)$ and $G(t)$. Even if the respective Hamiltonians H and $\hat{H}$ do not discriminate between particles and antiparticles (e.g. $H(t) = \hat{H}(t)$) there still is a difference due to the fact that neutrino-antineutrino coupling can be complex and not self-adjoint, i.e. $\widetilde{B}(t) \neq \widetilde{B}^\dagger(t)$ in general.

A similar situation pertains to $\nu_a \to \nu_b$ and $\bar{\nu}_a \to \bar{\nu}_b$ oscillations. They differ in the time evolution operators for neutrinos and antineutrinos, but also in the second order operators $\mathcal{O}^{(2)}(t)$ and $\hat{\mathcal{O}}^{(2)}(t)$. Those operators are, in general, not identical, which is again due to the fact that $\widetilde{B}(t)$ does not have to be self-adjoint. This statement translates to the fact that lepton number violating neutrino oscillations discriminate between particles and antiparticles even if there is no difference in the respective Hamiltonians for neutrinos and antineutrinos. This behavior is also expected from lepton number violation; note, that this distinguishing feature is generated dynamically rather than being imposed by hand.

Having established an approach to neutrino oscillations in which lepton number violation is considered a *small* effect, let us now elaborate on the physical notion underlying this framework. For such purposes we begin by studying the one generation case of neutrino oscillations, since already in this case lepton number violation allows neutrino-antineutrino oscillations to develop. Making use of the formalism, we obtain

$$\widetilde{B}(t, t_0) = \int_{t_0}^{t} \mathrm{d}\tau \; B(\tau)\, e^{i\Delta\widetilde{H}(\tau)}, \tag{43}$$

where we have defined

$$\Delta\widetilde{H}(t) \equiv \int_{t_0}^{t} \mathrm{d}\tau \; \Delta H(\tau) = \int_{t_0}^{t} \mathrm{d}\tau \left[H(\tau) - \hat{H}(\tau)\right] \tag{44}$$

as the difference between (CP non-conserving) *potential* terms in the neutrino and antineutrino sectors. Clearly, if the difference in such potential terms vanishes for some time $t = t_{\rm res}$, the integral Eq. (43) has a stationary phase and we can evaluate it by means of the saddle point approximation. This yields

$$\widetilde{B} \simeq e^{i\Delta\widetilde{H}(t_{\rm res})}\sqrt{\frac{\pi}{2}\frac{1}{\gamma_{\rm res}}}. \tag{45}$$

Here we have introduced the *adiabaticity parameter at resonance* $\gamma_{\rm res}$, which we shall properly define and explain shortly. To this end, let us start from the Hamiltonian of Eq. (2) and notice that for a simple two dimensional case, we can always easily find a unitary transformation (for a two dimensional problem phases are irrelevant and the unitary transformation amounts to a time-dependent rotation in flavor space), which entails an *effective mixing angle* $\Theta(t)$ fixed via

$$\cos\Theta(t) = \frac{\Delta H(t)}{\omega_{\rm eff}}, \qquad \sin\Theta(t) = \frac{2|B(t)|}{\omega_{\rm eff}}. \tag{46}$$

The *effective oscillation frequency* $\omega_{\rm eff}$ of the system is here given by

$$\omega_{\rm eff} = \sqrt{4|B(t)|^2 + \Delta H^2(t)}. \tag{47}$$

It is interesting to note that the effective mixing becomes maximal, neutrino conversions undergo a resonance, in the case of vanishing $\Delta H(t)$, i.e. $\Delta H(t_{\rm res}) = 0$ at some resonance time $t_{\rm res}$. Put another way, mixing between neutrinos and antineutrinos becomes maximal when the difference between potential terms for the two species is minimal (or in fact vanishing at the resonance time). This is also corroborated by the fact that for identical neutrino and antineutrino potentials mixing is always maximal. Obviously, for the case of CP conserving neutrino and antineutrino potentials the difference between those vanishes at all times since they are identical to begin with. However, in the case of CP non-conserving matter potentials, the difference is generally non-zero and can be varying with time also — the latter effect is crucial for some physics implications of the resonance structure as shall be seen in due course. It is also conceivable to allow for CPT violating potential terms, which then generate a non-vanishing difference $\Delta H(t)$.

With the effective mixing at our disposal, we write the adiabaticity parameter at resonance as

$$\gamma(t_{\rm res}) \equiv \gamma_{\rm res} = \frac{1}{\omega_{\rm eff}} \frac{{\rm d}\Theta(t)}{{\rm d}t}\bigg|_{t=t_{\rm res}} = -\frac{1}{4|B(t)|^2} \frac{{\rm d}\Delta H(t)}{{\rm d}t}\bigg|_{t=t_{\rm res}} . \tag{48}$$

This definition of an adiabaticity parameter can also be understood physically as follows: the characteristic time of the neutrino-antineutrino system is $\tau_{\rm sys} \sim 1/\omega_{\rm eff}$, whereas the characteristic time of the interaction can be given as $\tau_{\rm int} \sim ({\rm d}\Theta/{\rm d}t)^{-1}$. Hence a small $\gamma_{\rm res}$ states that the system's time scale is much smaller than the interaction's time scale. Put another way, the transition is adiabatic.

For $\gamma_{\rm res} \gg 1$ we encounter non-adiabatic neutrino-antineutrino conversions; $\gamma_{\rm res} \ll 1$ gives the adiabatic case. So roughly speaking (neglecting the derivative of the potential term for the time being) the smaller $B(t)$, the larger the non-adiabaticity of the neutrino-antineutrino system and hence if we start the evolution with only ν_a ($\bar{\nu}_a$) states present the transition $\nu_a \to \nu_a$ ($\bar{\nu}_a \to \bar{\nu}_a$) prevails; the lepton number violating oscillation channel $\nu_a \to \bar{\nu}_a$ ($\bar{\nu}_a \to \nu_a$) gets more and more suppressed as the non-adiabaticity increases. If, however, the change of the difference in matter potentials is small with time, this effect can partially compensate a small neutrino-antineutrino coupling, driving the evolution of the system towards adiabatic transitions opening the lepton number violating oscillation channel $\nu_a \to \bar{\nu}_a$ ($\bar{\nu}_a \to \nu_a$) again. Note also that the suppression of the lepton number violating oscillation channel depends on the time dependence of the potential terms in the Hamiltonian.

Substituting the result for the adiabaticity parameter $\gamma_{\rm res}$ of Eq. (48) in the expression for $\widetilde{B}$ of Eq. (45) and at the same time keeping in mind that the oscillation probability $P(\nu_a \to \bar{\nu}_a; t)$ of Eq. (37) is directly proportional to $\widetilde{B}^\dagger$, it is seen that interpreting lepton number violating (neutrino-antineutrino) couplings as a small perturbation to lepton number conserving neutrino oscillations in the one generation case is equivalent to assuming that neutrino-antineutrino oscillations occur non-adiabatically; non-adiabaticity hence *closes* the lepton number violating oscillation channel $\nu_a \to \bar{\nu}_a$, but at the same time improves the perturbative expansion as outlined above.

These results for the one generation framework make the case for referring to the perturbation theory as developed in this letter as a *non-adiabatic perturbation expansion*.

From the definition of the adiabaticity parameter $\gamma_{\rm res}$ in Eq. (48) another interesting feature emerges. Suppose we assume *small* lepton number violating couplings $B(t)$ between neutrinos and antineutrinos in the oscillation Hamiltonian. This means that oscillations between particles and antiparticles are suppressed by the large non-adiabaticity of the transitions giving rise mostly to the lepton number conserving oscillation channel $\nu_a \to \nu_a$. If, however, the time variation of the difference in matter potentials at the resonance is sufficiently mild, it is seen that the presence of such matter along the neutrino (antineutrino) propagation path can drive the system towards adiabaticity thus opening the oscillation channel $\nu_a \to \bar{\nu}_a$. Put another way, the presence of CP (or even CPT) non-conserving matter of a varying density (readily re-interpreted as a time dependence of the potential terms) can enhance lepton number violating neutrino oscillations as compared to the case in vacuo. Note that this statement holds regardless of the nature of the perturbation theory developed in this letter, since it is merely a result obtained from the adiabaticity parameter $\gamma_{\rm res}$ and the latter does only need the Hamiltonian of the system as a prerequisite.

Let us next approach the two generation case in vacuo. In this case, apart from a term proportional to the unit matrix, the flavor oscillation of neutrinos as well as antineutrinos is governed by the Hamiltonian

$$H = \hat{H} = \frac{\Delta m^2}{4p} \Big(\sigma_x \sin 2\theta - \sigma_z \cos 2\theta \Big) \equiv \omega \Big(\sigma_x \sin 2\theta - \sigma_z \cos 2\theta \Big) , \tag{49}$$

where the σ's are the usual Pauli matrices, and θ is the mixing angle in absence of lepton number violating terms. Accordingly, the evolution operator in the neutrino as well as the antineutrino sector will be given by

$$G(t, t_0 = 0) = \hat{G}(t, t_0 = 0) = \cos \omega t - i \Big(\sigma_x \sin 2\theta - \sigma_z \cos 2\theta \Big) \sin \omega t , \tag{50}$$

in absence of lepton number violating terms. In order to include the effects of the lepton number violating terms, let us first write the matrix $B(t)$ as

$$B(t) = b_0(t) + b_i(t) \sigma_i , \tag{51}$$

where

$$b_0(t) = \frac{1}{2} \operatorname{tr} B(t) , \qquad b_i(t) = \frac{1}{2} \operatorname{tr} \Big(\sigma_i B(t) \Big) . \tag{52}$$

Note that if CPT is conserved, Eq. (7) dictates that we must have $b_y(t) = 0$. It is straightforward to see that

$$\dot{\widetilde{B}}(t) = \beta_0(t) + \beta_i(t) \sigma_i , \tag{53}$$

where

$$\begin{aligned}
\beta_0(t) &= b_0(t) , \\
\beta_x(t) &= \sin 2\theta \Big[b_x(t) \sin 2\theta - b_z(t) \cos 2\theta \Big] + \cos 2\omega t \cos 2\theta \Big[b_x(t) \cos 2\theta + b_z(t) \sin 2\theta \Big] \\
&\quad - b_y(t) \sin 2\omega t \cos 2\theta , \\
\beta_y(t) &= b_y(t) \cos 2\omega t + \sin 2\omega t \Big[b_x(t) \cos 2\theta + b_z(t) \sin 2\theta \Big] , \\
\beta_z(t) &= - \cos 2\theta \Big[b_x(t) \sin 2\theta - b_z(t) \cos 2\theta \Big] + \cos 2\omega t \sin 2\theta \Big[b_x(t) \cos 2\theta + b_z(t) \sin 2\theta \Big] \\
&\quad - b_y(t) \sin 2\omega t \sin 2\theta .
\end{aligned} \tag{54}$$

We will assume here that the elements of B are time-independent. These expressions can then be easily integrated to give $\widetilde{B}$, and the resulting expressions plugged into the relevant formulas to obtain various oscillation probabilities.

For the most general B, the expressions are cumbrous. Here, we present an illustrative example for a very simple form of B which can be analytically tackled without much trouble. Consider B to be the multiple of a unit matrix, which means that the lepton number violating effects are somehow flavor blind. In Eq. (51), this would mean that all $b_i = 0$, only b_0 is non-zero. Now, from Eq. (54), we find that $\widetilde{B}$ is $b_0 t$ times the unit matrix. Looking back at Eqs. (37) and (38), we find that the probabilities of neutrino-antineutrino oscillations are given by

$$P(\nu_a \to \bar{\nu}_b; t) = |b_0|^2 t^2 \times P_0(\nu_a \to \nu_b; t) , \tag{55}$$

whereas the lepton number conserving oscillation probabilities are given by

$$P(\nu_a \to \nu_b; t) = \Big(1 - |b_0|^2 t^2 \Big) \times P_0(\nu_a \to \nu_b; t) . \tag{56}$$

Let us discuss the nature of this solution. If there is no lepton number violation, a plot of oscillation probability vs. t would exhibit periodic highs and lows of the same magnitude. If there is lepton number violation, these periodic nature would be modulated by the function $(1 - |b_0|^2 t^2)$. This means that, as the path length increases, the successive maxima and minima of probability would be less and less pronounced. There is, of course, no fear of the maxima and minima vanishing and the curve becoming completely flat, because, as said earlier, these solutions should be valid in the region where $|b_0|t \ll 1$.

In fact, with this particular choice of the matrix B, one can do better and solve the problem exactly. To see this, let us recall the form of the interaction Hamiltonian given in Eq. (18). Since $G = \hat{G}$ for the Hamiltonian $\mathbb{H}_0$, we obtain the 2×2 block structure of $\mathbb{H}_\mathrm{I}$ as follows:

$$\mathbb{H}_\mathrm{I} = \begin{pmatrix} 0 & b_0 \\ b_0^* & 0 \end{pmatrix} = |b_0| \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}, \tag{57}$$

where $b_0 = |b_0|e^{i\alpha}$. Moreover, in the exponent in the Magnus expansion, only the term Ω_1 is non-zero. From Eq. (23), we find that it is given by

$$\Omega(t) = \Omega_1(t) = -i\mathbb{H}_\mathrm{I} t\,. \tag{58}$$

Because of the special form for the matrix $\mathbb{H}_\mathrm{I}$, the exponentiation can be done exactly, and one obtains

$$e^{\Omega(t)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos|b_0|t - i \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix} \sin|b_0|t\,. \tag{59}$$

The evolution operator of the system is defined through Eqs. (15) and (21). It shows that

$$\begin{aligned} P(\nu_a \to \nu_b; t) &= \cos^2|b_0|t \times P_0(\nu_a \to \nu_b; t)\,, \\ P(\nu_a \to \bar{\nu}_b; t) &= \sin^2|b_0|t \times P_0(\nu_a \to \nu_b; t)\,. \end{aligned} \tag{60}$$

The expressions given in Eqs. (55) and (56) are nothing but the approximations up to leading order terms in the time dependence. However, in the form given here in Eq. (60), it is valid for all t. Note that, although this last illustration uses a time-independent Hamiltonian, the general formulas that we give in Eqs. (35–38) are valid for time-dependent cases as well, e.g., in matter induced oscillations involving neutrinos and antineutrinos.

In conclusion, we have developed a practical and efficient way of dealing with lepton number violating neutrino (antineutrino) oscillations. In a *non-adiabatic* perturbation theory, in which lepton number violation is treated as a small effect compared to common flavor oscillations, as is compatible with experimental findings, we have given explicit expressions for the various oscillation probabilities up to second order in the perturbation and have embarked upon their interpretation using illustrative examples for one and two neutrino generations. For the case of one neutrino generation, we have found a novel resonance in lepton number violating oscillations, which in CP non-conserving matter enhances a potentially small lepton number violation in vacuo. For the exactly solvable two generation case of time-independent and flavor blind lepton number violation in vacuo, we understand that in the presence of lepton number non-conservation the common flavor oscillation probabilities are periodically modulated with a frequency given by the lepton number violating coefficient.

Acknowledgements: SH is grateful to P.B. Pal for inviting him to spend two months at the Saha Institute of Nuclear Physics, Kolkata, as well as for his warm hospitality during this period. The bulk of this work was done during this stay in Kolkata. OM was supported by UEFISCDI project PN-II-RU-TE-2011-3-0184, no. 9/05/10/2011.

References

[1] C. Amsler *et al.* [Particle Data Group Collaboration], Phys. Lett. **B667**, 1-1340 (2008).

[2] A. Aguilar *et al.* [LSND Collaboration], Phys. Rev. **D64**, 112007 (2001). [hep-ex/0104049].

[3] A. A. Aguilar-Arevalo *et al.* [The MiniBooNE Collaboration], Phys. Rev. Lett. **98**, 231801 (2007). [arXiv:0704.1500 [hep-ex]],

A. A. Aguilar-Arevalo *et al.* [MiniBooNE Collaboration], Phys. Rev. Lett. **102**, 101802 (2009). [arXiv:0812.2243 [hep-ex]].

[4] A. A. Aguilar-Arevalo *et al.* [The MiniBooNE Collaboration], Phys. Rev. Lett. **105**, 181801 (2010). [arXiv:1007.1150 [hep-ex]].

[5] A. E. Nelson, Phys. Rev. **D84**, 053001 (2011). [arXiv:1010.3970 [hep-ph]].

[6] H. Päs, S. Pakvasa, T. J. Weiler, Phys. Rev. **D72**, 095017 (2005). [hep-ph/0504096],

S. Hollenberg, O. Micu, H. Päs, T. J. Weiler, Phys. Rev. **D80**, 093005 (2009). [arXiv:0906.0150 [hep-ph]].

[7] V. A. Kostelecky, M. Mewes, Phys. Rev. **D69**, 016005 (2004). [hep-ph/0309025],

S. Hollenberg, O. Micu, H. Päs, Phys. Rev. **D80**, 053010 (2009). [arXiv:0906.5072 [hep-ph]].

[8] For an early account see, e.g., D. -d. Wu, Phys. Lett. **B96**, 311 (1980).

[9] L. Wolfenstein, Phys. Rev. **D17**, 2369–2374 (1978).

[10] For a different kind of perturbation theory see, e.g., J. S. Diaz, V. A. Kostelecky, M. Mewes, Phys. Rev. **D80**, 076007 (2009). [arXiv:0908.1401 [hep-ph]].

[11] W. Magnus, Commun. Pure Appl. Math. **VII**, 649 (1954),

S. Blanes, F. Casas, J. A. Oteo, J. Ros, Phys. Rep. **470**, 151 (2009) and references therein

[12] S. Klarsfeld, J. A. Oteo, Phys. Rev. **A 39**, 3270 (1989).