SHORT INTRODUCTION TO GENERAL RELATIVITY

Ioana Duțan

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Contents:

I Introduction

→ in Newtonian physics the law were covariant under the 3D orthogonal group of transformations of space coordinates which preserve :

$$r^2 = x^2 + y^2 + z^2$$

→ the gravitational force between 2 masses is covariant this orthogonal group $(O \cdot O^T = I)$ as well as the Poisson equation :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \equiv \nabla^2 \phi = -4\pi G \rho$$

→ in SR , the orthogonal group is replaced by the Lorentz group which preserves

$$s^2 = c^2 t^2 - x^2 - y^2 - z^2$$

→ the generalization of the Poisson eq :

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi \equiv \Box \phi = 4\pi G \rho ,$$

! implies the propagation of the gravitational disturbance with the speed of light

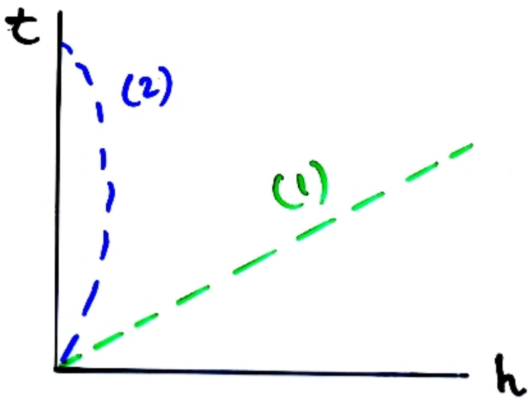
→ Incompatibilities between Newtonian Gravitation, SR, and Gravity

- → Maxwell equations (Lorentz invariant) are not invariant under the Galilei group
- → generalization of the Poisson eq. (in SR) does not show how the light is affected by gravitation
- → incompatibility of the gravitational phenomena with the concept of the inertial frame.

→ Einstein identified the intrinsic property of the spacetime with its geometry

Gravitation $\equiv$ Spacetime geometry

→ in the presence of a source of gravitation, the gravitational effects will not be described through an explicit external force but through the non-Euclidean nature of the spacetime geometry!



(1) trajectory of a freely moving particle

(2) particle moving under the Earth's gravity

- → in Newtonian gravity, it's the Earth gravitational FORCE which bends the trajectory
- → in Einstein's view the line is also "straight" but the spacetime is NON-EUCLIDEAN because of Earth's gravity

II Vectors & Tensors

1. Scalars
2. Vectors
   a) contravariant vectors
   b) covariant vectors
3. Tensors
   a) metric tensor
   b) Levi-Civita tensor
   c) Dual tensor

II Vectors & Tensors

→ in SR : rectangular Cartesian coordinate system

$x^0 = ct \,,\; x^1 = x \,,\; x^2 = y \,,\; x^3 = z$

$x^\mu \; (\mu = 0,1,2,3)$

- line-element :

$$\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu}$$

↳ infinitesimal coordinate changes between $P(x^\mu)$ and $P'(x^\mu + dx^\mu)$

- $\eta_{\mu\nu} = \begin{bmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$, with signature -2 !

(Minkowski spacetime)

→ non-Euclidean spacetime of GR will also be described by 4 general coordinates x^μ.

- coordinate patches = subregions of the spacetime which can be covered be one coordinate system

e.g. : The surface of the unit sphere $x^2+y^2+z^2=1$ can be covered by 6 coordinate patches. In one such patch (~~y,z~~) serve as coordinates for $x>0$, $-1<y<1$, $-1<z<1$.
(y,z)

- line-element (spacetime metric)

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

↓

metric functions ⇒ max 10 independent m.f. since $g_{\mu\nu}$ is symmetric

- Coordinate transformations : $x^\mu \to x'^\mu$

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \quad \& \quad dx^\nu = \frac{\partial x^\nu}{\partial x'^\mu} dx'^\mu$$

and $$\frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\rho} = \delta^\mu_\rho = \begin{cases} 1, & \mu = \rho \\ 0, & \text{otherwise} \end{cases}$$ (Kronecker delta)

→ non-singularity of transformation $\Leftrightarrow$ non-singularity of the matrix $\left\| \frac{\partial x'^\mu}{\partial x^\nu} \right\|$

→ quantities related to the intrinsic properties of the spacetime should not depend on any specific coordinate system !
quantities = scalars, vectors, & tensors

① Scalars

→ quantities which do not change their magnitude under a coordinate transformation (invariants of the transformation)

→ consider a scalar field ϕ, then:

$$\phi[x^\mu] = \phi[x^\mu(x'^\rho)] = \phi'(x'^\rho)$$

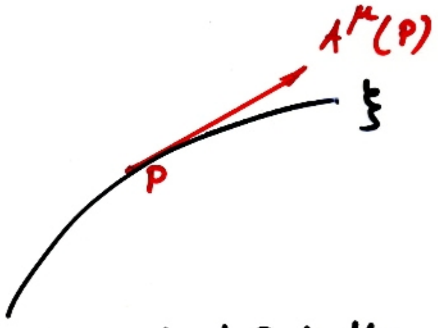
→ ds^2 : should remain unchanged under a coordinate transf.

$$g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$$

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$$

(2) Vectors

a) Contravariant vectors



The tangent at P to the curve ξ is a contravar. vector

→ consider first the tangents to curve in spacetime

→ parametrization of the curve ξ is the functions $x^\mu(\lambda)$
λ = parameter along ξ.

$$A^\mu(P) = \left.\frac{dx^\mu}{d\lambda}\right|_P$$

, the components of the tangent vector to ξ at point P.

→ a curve and its tangent $\neq f$(choice of coordinate system) !
⇒ in another coordinate system we can define $A'^\mu(P)$ in a similar way

→ there exists a linear relation at P between A^μ and A'^μ

$$\boxed{A'^\mu(P) = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_P A^\nu(P)}$$

, def. for contravariant vectors

→ they transform in the same way as coordinates.

→ contravariant vector field = a set of functions $A^\mu(x^\nu)$ of coordinates which transforms as a contravariant vector at every point (x^ν) of spacetime.

b) Covariant vectors

→ consider a scalar field $\phi(x^\nu)$

→ $\phi(x^\nu) = C$ ≡ series of surface Σ in spacetime

→ define the normal to the Σ-surface through a point P as a vector:

$$B_\mu(P) \equiv \left.\frac{\partial\phi}{\partial x^\mu}\right|_P$$

→ $x^\mu \to x'^\mu$ (a normal should be coordinate-independent):

$$\boxed{B'_\mu(P) = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) B_\nu(P)}$$

, def. for covariant vectors

→ dynamical analogy:

- → velocity (tangent) like vectors = contravariant
- → force (normal) like vectors = covariant
- → rate of work = scalar

→ $A^\mu B_\mu$ = scalar ⇒ invariant

→ $x^\mu \to x'^\mu$: $A'^\mu B'_\mu = \frac{\partial x'^\mu}{\partial x^\rho} A^\rho \frac{\partial x^\sigma}{\partial x'^\mu} B_\sigma = \delta_\rho^{\ \sigma} A^\rho B_\sigma \overset{\sigma=\rho\,\hat{=}\,\mu}{=} A^\mu B_\mu$

$$\boxed{\begin{aligned} A'^\mu A'^\nu &= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} A^\rho A^\sigma \\ B'_\mu B'_\nu &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} B_\rho B_\sigma \\ A'^\mu B'_\nu &= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} A^\rho B_\sigma \end{aligned}}$$

↳ these transformation laws lead to the definition of <u>tensors</u>

3 Tensors

→ a <u>contravariant tensor</u> of rank n = an object with 4^n components & the components are specified by n upper indices

$x^\mu \to x'^\mu$: $$T'^{\mu_1 \dots \mu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\nu_n}} T^{\nu_1 \dots \nu_n}$$

→ <u>covariant tensor of rank</u> n

$$T'_{\mu_1 \dots \mu_n} = \frac{\partial x^{\mu_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\mu_n}}{\partial x'^{\nu_n}} T_{\nu_1 \dots \nu_n}$$

! $B'_\mu B'_\nu$ and $g'_{\mu\nu}$ are second rank covariant tensors → (0,2)

→ <u>mixed tensor</u> : $A'^\mu B'_\mu$ and $\delta'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \delta^\rho{}_\sigma$

→ <u>contraction</u> : $T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}$, rank $m+n$

if say $\mu_1 = \nu_1$ ⇒ tensor rank $m+n-2$

= sum over the identified (dummy) index.

→ a scalar obtained by contracting a (0,2) tensor = trace of the tensor

→ <u>symmetric & antisymmetric tensors</u>

a) $S_{\mu\nu} = S_{\nu\mu}$; if a tensor is symmetric in one coord frame then it's symmetric in any coord frame !

$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ = symmetrization = to construct a symm tensor from any other tensors (same rank)

b) $A_{\mu\nu} = -A_{\nu\mu}$;

$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$ = antisymmetrization.

a) The metric tensor ; raising & lowering of indices

→ $g_{\mu\nu}$ = covariant second rank tensor (0,2)

→ introduce the inverse of the metric : $\boxed{g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu}}$

the 16 quantities transf. like a covariant vector

assuming $\boxed{g = \det g_{\mu\nu} \neq 0}$

$\boxed{g < 0}$ because of the signature (−2)

→ $\boxed{\det g}$ does not transform as a scalar ! : $x^{\mu} \to x'^{\mu}$

$$g' = \det g'_{\mu\nu} = \det \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma} \right) =$$

$$= \det \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}} \right) \det g_{\rho\sigma} \det \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) = \left[\underbrace{J\{x \to x'\}}_{\text{Jacobian of the transf } x \to x'} \right]^2 g$$

Since $dx^0 dx^1 dx^2 dx^3$ transforms as :

$$dx^0 dx^1 dx^2 dx^3 = J\{x \to x'\}\, dx'^0 dx'^1 dx'^2 dx'^3$$

⇒ $\sqrt{-g}\, dx^0 dx^1 dx^2 dx^3$ is an invariant under coord transf.

$\underbrace{dx^4 = d\Omega}_{\text{volume element}}$ ⇒ $\boxed{\sqrt{-g}\, d\Omega}$ invariant

→ raising & lowering = gymnastics of indices

$$\left.\begin{aligned} A_{\mu} &= g_{\mu\nu} A^{\nu} \\ B^{\mu} &= g^{\mu\nu} B_{\nu} \end{aligned}\right\}$$ for a contravariant vector there is a corresp. covariant vector (& vice versa)

$$T^{\dots\mu_r\dots\dots}_{\dots\dots\dots\nu_s\dots}\, g_{\mu_r\rho}\, g^{\nu_s\sigma} = T^{\dots\dots\sigma\dots}_{\dots\rho\dots\dots}$$

b) The Levi-Civita tensor

→ consider the totally antisymmetric symbol

$$(\mu\nu\rho\sigma) = \begin{cases} +1 , & \text{if } (\mu,\nu,\rho,\sigma) \text{ are even permutation of } (0,1,2,3) \\ -1 , & \text{" odd "} \\ 0 , & \text{otherwise} \end{cases}$$

→ define the Levi Civita tensor

$$\boxed{\begin{aligned} \epsilon_{\mu\nu\rho\sigma} &= (-g)^{1/2} [\mu\nu\rho\sigma] \\ \epsilon^{\mu\nu\rho\sigma} &= (-g)^{-1/2} [\mu\nu\rho\sigma] \end{aligned}}$$

→ importance of the LC tensor : can be used to construct duals of tensors

c) Dual tensor

→ defined by : $A^*_{\mu} = \epsilon_{\mu\nu\rho\sigma} A^{\nu\sigma\rho}$

$$A^*_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} A^{\sigma\rho}$$

→ used in electromagnetism

III Tensor calculus

1. Parallel transport
2. Covariant differentiation
3. Riemannian Affine Connection

III Tensor calculus

→ the laws of physics are usually described by means of differential equations, relating changes of physical quantities (vectors, tensors, etc) with respect to space & time.

→ we see next how their rates of change with respect to the spacetime coord x^μ can be described.

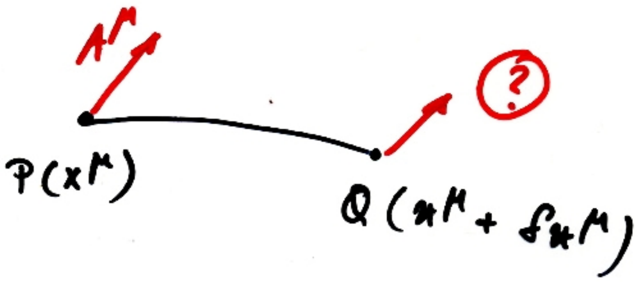
→ we know the derivative of a scalar field is a vector.
!?! Is the derivative of a vector field a tensor? NO!

$x^\mu \to x'^\mu$

$$A'^\mu{}_{,\nu} = \frac{\partial A'^\mu}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial}{\partial x^\rho}\left\{\frac{\partial x'^\mu}{\partial x^\sigma}A^\sigma\right\} =$$

$$= \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\sigma}A^\sigma{}_{,\rho} + \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\rho\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}A^\sigma}_{!}$$

① Parallel transport



→ !?! if we shift the vector from P to Q without changing its magnitude & direction, what are its components at Q?

→ just imagine that the changes $\boxed{\delta A^\mu}$ in the components of A are linear functions of δx^ρ and A^ν

$$\boxed{\delta A^\mu = -\Gamma^\mu_{\nu\rho} A^\nu \delta x^\rho}$$ → for a contravariant vector

↳ $4^3 = 64$ component entity

→ $\boxed{\Gamma^{\mu}_{\nu\rho}}$ = are functions of the spacetime coord x^{μ}
= called affine - connection on the spacetime region
(Christoffel symbols)

→ $\boxed{\delta B_{\nu}}$: changes in the components of a covariant vector under the parallel transport

$$0 = \delta(A^{\mu}B_{\mu}) = \delta A^{\mu} B_{\mu} + A^{\mu}\delta B_{\mu} = -\Gamma^{\mu}_{\nu\rho} A^{\nu}\delta x^{\rho} + A^{\mu}\delta B_{\mu}$$

$\overset{\forall A^{\mu}}{\Leftrightarrow}$ $A^{\nu}(\delta B_{\nu} - \Gamma^{\mu}_{\nu\rho} B_{\mu}\delta x^{\rho}) = 0$

$$\Rightarrow \boxed{\delta B_{\nu} = \Gamma^{\mu}_{\nu\rho} B_{\mu}\delta x^{\rho}}$$ → for a covariant vector

→ since $A_{\mu}A^{\mu}$ is unchanged under parallel transport ⇒ the magnitude of A is preserved

→ the rule for parallel transport of tensors

$$\delta T_{\mu\nu} = \Gamma^{\rho}_{\mu\sigma} T_{\rho\nu}\delta x^{\sigma} + \Gamma^{\rho}_{\nu\sigma} T_{\mu\rho}\delta x^{\sigma}$$

(2) Covariant differentiation

→ use of the parallel transport

→ the physical change in A^{μ} from P to Q is given not by

$$dA^{\mu} = \frac{\partial A^{\mu}}{\partial x^{\nu}}\delta x^{\nu} \quad \text{(to first order)}$$

but by:

$$DA^{\mu} = dA^{\mu} - \delta A^{\mu} = \left(\frac{\partial A^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\rho\nu} A^{\rho}\right)\delta x^{\nu}$$

↓

is the change coming from the parallel transport from P to Q

→ the change DA^μ should be coordinate-independent and hence transform as a vector

→ since δx^ν transform as the components of a contravariant vector, the combination :

$$\boxed{D_\nu A^\mu = A^\mu{}_{;\nu} \equiv \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\rho\nu} A^\rho}$$ behaves like a mixed tensor

↓

covariant derivative

$$\begin{cases} \phi_{;\nu} = \phi_{,\nu} & \rightarrow \text{for a scalar} \\ B_{\mu;\nu} = B_{\mu,\nu} - \Gamma^\rho_{\mu\nu} B_\rho & \rightarrow \text{for a covariant vector} \\ T_{\mu\rho;\nu} = T_{\mu\rho,\nu} - \Gamma^\sigma_{\mu\nu} T_{\sigma\rho} - \Gamma^\sigma_{\rho\nu} T_{\mu\sigma} & \rightarrow \text{for a tensor} \end{cases}$$

→ transformation rule for the affine-connections ($x^\mu \rightarrow x'^\mu$)

$$\Gamma^\mu_{\nu\rho} = \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\tau}{\partial x'^\rho} \Gamma'^\lambda_{\sigma\tau} + \underbrace{\frac{\partial^2 x'^\lambda}{\partial x^\nu \partial x^\rho} \frac{\partial x^\mu}{\partial x'^\lambda}}$$

! do not transform as a tensor

(3) Riemannian Affine Connection

→ impose further conditions on the 64 quantities $\Gamma^\mu_{\nu\rho}$:

(i) $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$

(ii) $g_{\mu\nu;\rho} = 0$

⇒ the affine connection is said to be Riemannian and the geometry is called the Riemannian geometry.

→ the Riemannian affine connection is specified completely by the spacetime metric & its first derivatives

$$\boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left[g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right]}$$

→ useful relations :

$$\begin{cases} \Gamma^{\mu}_{\nu\mu} = \dfrac{\partial}{\partial x^{\nu}} \left\{ \ln \sqrt{-g} \right\} \\ g^{\mu\nu} \Gamma^{\xi}_{\mu\nu} = - \dfrac{1}{\sqrt{-g}} \dfrac{\partial}{\partial x^{\rho}} \left\{ \sqrt{-g}\; g^{\rho\xi} \right\} \end{cases}$$

$$\begin{cases} A^{\mu}{}_{;\mu} = \dfrac{1}{\sqrt{-g}} \dfrac{\partial}{\partial x^{\mu}} \left(\sqrt{-g}\, A^{\mu} \right) \\ F^{[\mu\nu]}{}_{;\nu} = \dfrac{1}{\sqrt{-g}} \dfrac{\partial}{\partial x^{\nu}} \left(\sqrt{-g}\, F^{[\mu\nu]} \right) \end{cases}$$

IV Spacetime curvature

1. Geodesics
2. Riemann tensor
3. Bianchi identity
4. Ricci tensor
5. Curvature scalar

IV Spacetime curvature

→ the machinery of vectors, tensors & the affine connection permit us to quantify the intrinsic property which distinguishes between a flat & a curved spacetime. How to do that?

→ bad #1: in distinguishing between an Euclidean & a non-Euclidean spacetime we cannot rely on $g_{\mu\nu}$ alone

→ because a simple coordinate transf can produce apparently different $g_{\mu\nu}$

e.g: (1) $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 d\varphi^2)$ arrises from

(2) $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ of SR

→ bad #2: the Christoffel symbols (which involves $\partial_\rho g_{\mu\nu}$) do not help either → no components of a tensor → no covariance

C. symb = 0 for (2) and ≠ 0 for (1).

→ good: the desired distinguishing quantity involves $\partial^2 g$:

Riemann tensor - later

(1) Geodesics

→ straight line?

→ its direction does not change as we move along it

→ it represents the path of shortest distance between any 2 given points

Derivation of the geodesic equation

→ we start from the 3D case

→ the line-element is $ds^2 = g_{ik}\, dq^i dq^k$, q = generalized coordinate

→ in spherical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \Rightarrow g_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

→ the Lagrangean function is : $\mathcal{L} = T - V$

where $V = V(q^i)$ & $T = \frac{m}{2}\left(\frac{ds}{dt}\right)^2 = \frac{m}{2} g_{ik}\, \dot{q}^i \dot{q}^k$, $\dot{q}^i = \frac{dq^i}{dt}$

→ Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial q^i} \qquad (1)$$

$$(2)\quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} = \frac{m}{2}\frac{d}{dt}\left(2 g_{ik}\dot{q}^k\right)$$

← by renaming the indices

$$= m\frac{d}{dt}\left(g_{ik}\dot{q}^k\right) = m\left(\frac{\partial g_{ik}}{\partial q^j}\dot{q}^j\dot{q}^k + g_{ik}\ddot{q}^k\right)$$

$$(3)\quad \frac{\partial \mathcal{L}}{\partial q^i} = \frac{\partial T}{\partial q^i} - \frac{\partial V}{\partial q^i} = \frac{m}{2}\frac{\partial g_{ik}}{\partial q^i}\dot{q}^j\dot{q}^k - \frac{\partial V}{\partial q^i} \quad \Big/ \cdot \frac{1}{m}\sum_i g^{li}$$

(2) + (3) in (1)

$$\Rightarrow \ddot{q}^l + \underbrace{\left(g^{li}\frac{\partial g_{ik}}{\partial q^j} - \frac{1}{2} g^{li}\frac{\partial g_{ik}}{\partial q^i}\right)}_{(4)}\dot{q}^j\dot{q}^k = -\frac{1}{m} g^{li}\frac{\partial V}{\partial q^i}$$

→ evaluation of term (4) : $= \frac{1}{2} g^{li}\left(2\underbrace{\frac{\partial g_{ik}}{\partial q^j}} - \frac{\partial g_{ik}}{\partial q^i}\right)$

we can split it in 2 parts because the metric is symmetric in j, k ($g_{ik} = g_{ki}$) : $\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{ki}}{\partial q^j}$

$$\Rightarrow (4) = \underbrace{\frac{1}{2} g^{li} \left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^i} \right)}_{\text{it's exactly the christoffel symbol } \Gamma^l_{jk}}$$

$$\Rightarrow \underbrace{\ddot{q}^l}_{\text{acceleration}} + \underbrace{\Gamma^l_{jk} \dot{q}^j \dot{q}^k}_{\text{virtual force}} = -\frac{1}{m} \underbrace{g^{li} \frac{\partial V}{\partial q^i}}_{\text{force}}$$

virtual force (coming from the fact that the space is curved) – – Coriolis force

→ for a freely particle motion ⇒ geodesic motion:

$$\boxed{\ddot{q}^l + \Gamma^l_{jk} \dot{q}^j \dot{q}^k = 0}$$

→ in GR:

$$\boxed{\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0}$$

$$w^\mu = \frac{dx^\mu}{ds}, \quad \text{4-velocity}$$

$$\Rightarrow \left. \begin{array}{l} \underbrace{\dfrac{dw^\mu}{ds}}_{\dfrac{dw^\mu}{dx^\rho} \dfrac{dx^\rho}{ds} =} + \Gamma^\mu_{\nu\rho} w^\nu w^\rho = 0 \end{array} \right\} \Rightarrow \frac{\partial w^\mu}{\partial x^\rho} + \Gamma^\mu_{\nu\rho} w^\nu = 0$$

$$\Rightarrow \boxed{D_\rho w^\mu = 0}, \quad \text{geodesic motion}$$

e.g. :

Let's calculate the radial null geodesic from $r=0, t=0$ in the spacetime with line-element :

$$(1) \qquad ds^2 = dt^2 - e^{2Ht}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \quad H = \text{const}$$

result

Radial null geodesic : the radial character is preserved if $\theta = \text{const}$, and $\phi = \text{const}$

From (1), the metric coefficients are :

$$g_{\mu\nu} = \begin{array}{c} \\ t \\ r \\ \theta \\ \varphi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \varphi \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{2Ht} & 0 & 0 \\ 0 & 0 & -e^{2Ht} r^2 & 0 \\ 0 & 0 & 0 & -e^{2Ht} r^2 \sin^2\theta \end{pmatrix} \end{array} \qquad (2)$$

From the first integral of the geodesic, which has the general form :

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

we get (for θ coordinate) : $g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right) = 0$

$$\left.\begin{aligned} &\Rightarrow g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 = 0 \\ &\text{from (2)} \quad g_{\theta\theta} = -e^{2Ht} r^2 \neq 0 \end{aligned}\right\} \Rightarrow \frac{d\theta}{d\lambda} = 0 \Rightarrow \theta = \text{const}$$

Similar for ϕ coordinates :

$$\left.\begin{aligned} &g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \\ &g_{\phi\phi} = -e^{2Ht} r^2 \sin^2\theta \neq 0 \end{aligned}\right\} \Rightarrow \frac{d\phi}{d\lambda} = 0 \Rightarrow \phi = \text{const}$$

→ apply now the first integral to the t & r coordinates:

$$\Rightarrow g_{tt}\left(\frac{dt}{d\lambda}\right)^2 = 0 = g_{rr}\left(\frac{dr}{d\lambda}\right)^2$$

$$\Leftrightarrow 1\cdot\left(\frac{dt}{d\lambda}\right)^2 = e^{2Ht}\left(\frac{dr}{d\lambda}\right)^2 \qquad (5)$$

→ since $\left(\frac{d\phi}{d\lambda}\right) = \left(\frac{d\theta}{d\lambda}\right) = 0$; we need to consider only the t and r equations for the null geodesic, which has the general form: $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$

we take for indices μ, ν, ρ the coordinates t, r, so that

$$\frac{d^2t}{d\lambda^2} + \Gamma^t_{rr}\left(\frac{dr}{d\lambda}\right)^2 = 0 \qquad (4)$$

the Christoffel symbol, for this case, is:

$$\Gamma^t_{rr} = \frac{1}{2}g^{tt}\left(\cancelto{0}{g_{tr,r}} + \cancelto{0}{g_{rt,r}} - g_{rr,t}\right)$$

we do not have mixed coefficients $g_{tr} = g_{rt} = 0$

$$\Rightarrow \Gamma^t_{rr} = -\frac{1}{2}g^{tt}g_{rr,t} = -\frac{1}{2}\cdot 1\cdot\left(-2He^{2Ht}\right)$$

we use $r=0$, $t=0$ at the starting point of the geodesic

$$(4) \Rightarrow \left.\begin{aligned} &\frac{d^2t}{d\lambda^2} + He^{2Ht}\left(\frac{dr}{d\lambda}\right)^2 = 0 \\ &\text{from (5): } \left(\frac{dr}{d\lambda}\right)^2 = e^{-2Ht}\left(\frac{dt}{d\lambda}\right)^2 \end{aligned}\right\} \Rightarrow \frac{dt}{d\lambda} = \frac{1}{H(\lambda+\lambda_0)} \;\Big/\!\!\int_0^t$$

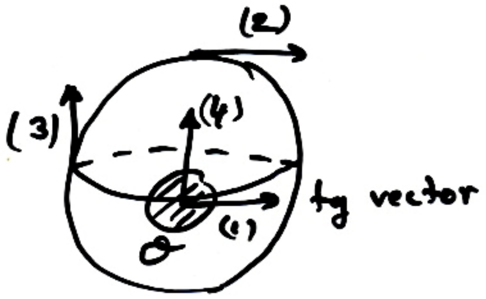
λ_0: ct of integr.

$$\text{so} \quad \left.\begin{aligned} &\Rightarrow t = \frac{1}{H}\ln\left(1+\frac{\lambda}{\lambda_0}\right) \\ &\text{and } r = \frac{1}{H}\frac{\lambda}{\lambda+\lambda_0} \end{aligned}\right\}$$

the coordinates t & r in terms of the affine parameter λ.

Derivation of the Riemann tensor (curvature tensor)

→ the change of vector under parallel transport (see the difference between the vectors at positions : (1) & (4)) around a closed curve is a measure of curvature !



→ we take the variation in the components of a vector under the parallel transport (see the slide with parallel transport):

$$\left.\begin{aligned} \delta V^\mu &= -\Gamma^\mu_{\nu\rho} V^\nu dx^\rho \\ \delta V_\mu &= +\Gamma^\sigma_{\mu\rho} V_\sigma dx^\rho \end{aligned}\right\}$$

because the changes are infinitesimal , the closed curve must be infinitesimal too. (in the region limited by ϑ)

$$\Rightarrow \Delta V_\mu = \int_{\partial\vartheta} \Gamma^\sigma_{\mu\nu} V_\sigma dx^\nu \qquad (1)$$

→ use of Stokes theorem :

$$\int_{\partial\vartheta} A_\mu \underbrace{dx^\mu}_{\text{line-element}} = \int_{\vartheta} \underbrace{df^{\mu\nu}}_{\substack{\text{volum}\\\text{elem}}} \frac{\partial A_\mu}{\partial x^\nu} \xrightarrow{\text{antisymmetrisation}} = \frac{1}{2}\int df^{\mu\nu}\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right) \quad (2)$$

from (1) & (2) $\Rightarrow \Delta V_\mu = \frac{1}{2}\Delta f^{\nu\rho}\left[\frac{\partial}{\partial x^\nu}\left(\Gamma^\sigma_{\mu\rho} V_\sigma\right) - \frac{\partial}{\partial x^\rho}\left(\Gamma^\sigma_{\mu\nu} V_\sigma\right)\right] \Leftrightarrow$

$$\Leftrightarrow \Delta V_\mu = \frac{1}{2}\Delta f^{\nu\rho}\left[\partial_\nu\left(\Gamma^\sigma_{\mu\rho}\right) V_\sigma + \Gamma^\sigma_{\mu\rho}\partial_\nu V_\sigma - \partial_\rho\left(\Gamma^\sigma_{\mu\nu}\right)V_\sigma - \Gamma^\sigma_{\mu\nu}\partial_\rho V_\sigma\right]$$

but $\frac{\partial V_\mu}{\partial x^\rho} = \Gamma^\sigma_{\mu\rho} V_\sigma$

$\Rightarrow$

$$\Rightarrow \Delta V_\mu = \frac{1}{2}\Delta f^{\nu\rho}\Big\{\left[\partial_\nu\left(\Gamma^{\sigma}_{\mu\rho}\right) - \partial_\rho\left(\Gamma^{\sigma}_{\mu\nu}\right)\right]V_\sigma +$$

(3)

$$+\Gamma^{\sigma}_{\mu\rho}\Gamma^{\gamma}_{\sigma\nu}V_{\gamma\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\gamma}_{\sigma\rho}V_{\gamma}\Big\}$$

α, α, σ, σ

rename the indices : $\sigma \leftrightarrow \alpha$, $\gamma \leftrightarrow \sigma$; $\gamma \leftrightarrow \sigma$

$$\Leftrightarrow \Delta V_\mu = \frac{1}{2}\Delta f^{\nu\rho} V_\sigma \cdot \left\{ R^{\sigma}_{\mu\nu\rho} \right\}$$

↳ what is left after getting the factor V_σ out of $\{\ \}$ in (3)

$$\Rightarrow \boxed{R^{\sigma}_{\mu\nu\rho} = \partial_\nu\Gamma^{\sigma}_{\mu\rho} - \partial_\rho\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\alpha\rho}}$$

Riemann tensor !

Properties of $R^{\mu}{}_{\nu\rho\sigma}$

→ $R^{\mu}{}_{\nu\rho\sigma} = 0 \Rightarrow$ space is Euclidean (flat)

→ locally flat spaces : $\Gamma^{\mu}{}_{\nu\rho}(x_{f}) = 0$, only in that point; in its vecinity $\neq 0$

but $R^{\mu}{}_{\nu\rho\sigma}$ will not vanish at this point

→ $R^{\sigma}{}_{\mu\nu\rho} = -R^{\sigma}{}_{\mu\rho\nu}$ → antisymmetric in the last 2 indices

→ $R^{\sigma}{}_{\mu\nu\rho} + R^{\sigma}{}_{\nu\rho\mu} + R^{\sigma}{}_{\rho\mu\nu} = 0$

↓ has 4^4 components $\xrightarrow[\text{symmetry}]{\text{by its}}$ 20 components $\xrightarrow[\text{symmetry}]{\text{by }\Gamma\text{'s}}$ 14 components

→ symmetry properties : $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$

$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$

$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$

Bianchi identity

→ $R^{\mu}{}_{\underbrace{\nu\rho\sigma;\varphi}_{\text{cyclic change}}} + R^{\mu}{}_{\nu\varphi\rho;\sigma} + R^{\mu}{}_{\nu\sigma\varphi;\rho} = 0$

Ricci tensor = trace of the Riemann tensor

→ $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = R^{\sigma}{}_{\mu\sigma\nu}$

$R_{\mu\nu} = R_{\nu\mu}$

→ if $R_{\mu\nu} = 0 \Rightarrow$ Ricci flat

the trace $= 0$ if the tensor = antisymmetric

Curvature scalar

→ $R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$

V Einstein's Equations from an Action Principle

→ use of the Hilbert action principle

→ we need a suitable scalar to be used as the Lagrangian

→ the simplest one : scalar curvature R

→ take the variation of the integral : $I = \int_{\mathcal{V}} R\sqrt{-g}\, d^4x$
over the spacetime $\mathcal{V}$ with a bounding 3-surface Σ

→ the variation is with respect to the change

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \quad \& \quad g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$$

where $\delta g_{\mu\nu}$ and $\delta g_{\mu\nu,\rho}$, etc , vanish on Σ !

→ at any point P in $\mathcal{V}$:

$$\delta(R\sqrt{-g}) = \delta(R_{\mu\nu} g^{\mu\nu}\sqrt{-g}) =$$

$$= \delta R_{\mu\nu} \cdot g^{\mu\nu}\sqrt{-g} + R_{\mu\nu}\, \delta(g^{\mu\nu}\sqrt{-g})$$

→ from $dg = g \cdot g^{\mu\nu} dg_{\mu\nu} \overset{\text{should be determinated}}{\Rightarrow} \delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$ ⇒

differential of a determinant

$$\Rightarrow \delta(R\sqrt{-g}) = \delta R_{\mu\nu} \cdot g^{\mu\nu}\sqrt{-g} + \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\delta g^{\mu\nu}\sqrt{-g}$$

→ choose locally inertial coord at P so that :

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\rho}_{\mu\nu,\rho}$$

$$\Rightarrow g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\,\delta\left(\Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\rho}_{\mu\nu,\rho}\right) =$$

$$= g^{\mu\nu}\left(\delta\Gamma^{\rho}_{\mu\rho,\nu} - \delta\Gamma^{\rho}_{\mu\nu,\rho}\right) = W^{\nu}{}_{,\nu}$$

where $w^{\nu} = g^{\mu\nu}\,\delta\Gamma^{\rho}_{\mu\rho} - g^{\mu\rho}\,\delta\Gamma^{\nu}_{\mu\rho}$ (*)

→ $\Gamma^{\rho}_{\mu\rho}$ does not transform like a tensor but $\delta\Gamma^{\rho}_{\mu\rho}$ does !

$$(\delta\Gamma^{\mu}_{\nu\rho})' = \frac{\partial x'^{\mu}}{\partial x^{\sigma}}\,\frac{\partial x^{\tau}}{\partial x'^{\nu}}\,\frac{\partial x^{\lambda}}{\partial x'^{\rho}}\,(\delta\Gamma^{\sigma}_{\tau\lambda})$$

⇒ w^{ν} is a vector

⇒ $g^{\mu\nu}\,\delta R_{\mu\nu}$ is a scalar

→ we can make $w^{\nu}{}_{,\nu}$ a scalar by replacing ',' with ';'

so $g^{\mu\nu}\,\delta R_{\mu\nu} = w^{\nu}{}_{;\nu}$ → in the locally flat coord system at P

→ since it's a scalar relation, it must hold in all coord systems

→ can calculate:

$$\delta I = \int_{\mathcal{V}} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right)\sqrt{-g}\,\delta g^{\mu\nu}\,d^4x + \int_{\mathcal{V}} w^{\nu}{}_{;\nu}\sqrt{-g}\,d^4x$$

but $\displaystyle\int_{\mathcal{V}} w^{\nu}{}_{;\nu}\sqrt{-g}\,d^4x = \int_{\mathcal{V}} \frac{\partial}{\partial x^{\nu}}\left(w^{\nu}\sqrt{-g}\right)d^4x = \int_{\Sigma} w^{\nu}\,d\Sigma_{\nu} = 0$

$w^{\nu} = 0$ on Σ → (*) + b.c.

$$\Rightarrow \delta I = \int_{\mathcal{V}} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right)\sqrt{-g}\,\delta g^{\mu\nu}\,d^4x$$

→ action principle for the gravitational field in the presence of matter & energy described by the Lagrangian $\mathcal{L}$:

$$\mathcal{A} = \frac{1}{2\kappa}\int_{\mathcal{V}} R\sqrt{-g}\,d^4x + \int_{\mathcal{V}} \mathcal{L}\sqrt{-g}\,d^4x$$

coupling constant between the geometry of a spacetime & its matter content

→ the variation : $\frac{\delta \mathcal{A}}{\partial g^{\mu\nu}} = 0$

$\Rightarrow$ $\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k\, T_{\mu\nu}}$, Einstein's field equations

→ for week feelds : $g_{00} \simeq 1 - \frac{2\phi}{c^2}$

ϕ = Newtonian potential

$\boxed{k = \frac{8\pi G}{c^4}}$, coupling constant is determined in the week field approximation

→ for cosmology ! : An important generalization of the action is the addition of a cosmological constant term

$$\frac{1}{2k} \int_{\mathcal{V}} (R - 2\Lambda) \sqrt{-g}\, d^4x$$

p.s. The field equations are $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = + k T_{\mu\nu}$ for the metric signature "$- + + +$".