(General) **Relativistic (Magneto)hydrodynamics**

- a (G)R (M)HD code is used to compute the flow of gas around strong field gravity sources
- used to study supernova collapse and formation of BH, BH-BH binaries, NS-NS binaries, pulsar wind nebulae, accretion disks, *relativistic jets* from AGN and microquasars, etc.
Introductory remarks

- **Propose of this material**
  - to introduce or review some aspects of GR
  - to introduce the 3+1 decomposition of the spacetime (in the Eulerian formulation)
  - to provide a small derivation of the conservative systems of the hyperbolic PDE of the GRHD

- **Notes on the level of this material**
  - if it is too easy, just treat it as review, perhaps from a different perspective
  - if it is too fuzzy, concentrate on the concepts...
General relativity background

- **metric element:**
  \[ ds^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu \]

- for a flat spacetime of special relativity:
  \[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \]

- \( ds^2 < 0 \), interval is timelike; \( ds^2 > 0 \), interval is spacelike; \( ds^2 = 0 \), it is null

- 1D curve \( x^\mu(\lambda) \) in spacetime describes a series of events

- timelike curve (worldline) is parameterized by the proper time \( \tau \)
4-velocity is defined as: \( u^\mu = \frac{dx^\mu(\tau)}{d\tau} \)

in SR, the 4-velocity components are

\[
u^\mu = (u^0, u^1, u^2, u^3) = (W, W \vec{v}), \text{ where } W = \frac{1}{\sqrt{1 - \vec{v}^2}}\]

imagine 2 particles with worldlines that meet at point \( A \), having 4-velocity \( u_\mu \) and \( v^\mu \); their product is an invariant (it can be evaluated in an arbitrary reference frame)(Fig. 1)

in particular, in the Lorentz reference frame comoving with \( u^\mu \) you have

\[
u'_\mu = (-1, 0, 0, 0), \text{ and then } u_\mu v^\mu = u'_\mu v'^\mu = -v'^0 = -W\]
General relativity background

Fig. 1. Worldlines intersection
3+1 Eulerian formulation

3+1 decomposition

- Spacetime is foliated into a set of non-intersecting spacelike hypersurfaces, parameterized by a parameter usually called time $t$, s.t., the evolution between these surfaces is described by two kinematic variables.

- Metric can be written in a particular way:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Which in a component form is:

$$\begin{pmatrix}
g_{00} & g_{0j} 
g_{i0} & g_{ij}
\end{pmatrix} = \begin{pmatrix}
\beta_s \beta^s - \alpha^2 & \beta_j 
\beta_i & \gamma_{ij}
\end{pmatrix}, \quad \text{where } \gamma_{ij} = g_{ij}, \quad i, j = 1, 2, 3$$

$\gamma_{ij} = 3$-metric induced on each spacelike slice.
**3+1 Eulerian formulation**

- $\gamma_{ik}\gamma^{kj} = \delta^j_i$; gymnastics of indices $\beta_i = \gamma_{ij}\beta^j$

- Decomposition of the volume associated with the 4-metric into the volume associated with the 3-metric

$$\sqrt{-g} = \alpha\sqrt{\gamma}, \quad g = \det(g_{\mu\nu}), \quad \gamma = \det(\gamma_{ij})$$

- Let's consider 2 close spacelike hypersurfaces $\Sigma(t)$ and $\Sigma(t + dt)$ (Fig. 2)

- **Lapse function** $\alpha$ describes the rate of advance of time along a timelike unit vector normal to the hypersurface

- Spacelike **shift vectors** $\beta^i$ describe the motion of coordinates within a surface
3+1 Eulerian formulation

- 4-vector $n^\mu$ is the unit normal vector to the $\Sigma(t)$

$$n_\mu = (-\alpha, 0, 0, 0), \quad n^\mu = \left( \frac{1}{\alpha}, \frac{-\beta^i}{\alpha} \right)$$

- Eulerian observers are observers having $n$ as 4-velocity, at rest in the slice $\Sigma(t)$ and moving $\perp$ to this slice with clocks showing proper time

- i.e., the basis adapted to the Eulerian observer frames is:

$$e_{(\mu)} = \{n, \partial_i\}$$

- 4-vector $u^\mu$ is the 4-velocity of some particle
3+1 Eulerian formulation

Fig 2. Geometrical interpretation of $\alpha$, $p_i$. 
3+1 Eulerian formulation

Let’s translate the 4-velocity of a particle from the arbitrary coordinate frame \((S)\) to the Eulerian frame \((S')\).

<table>
<thead>
<tr>
<th>(P)</th>
<th>(S)</th>
<th>(S')</th>
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</thead>
<tbody>
<tr>
<td>((t, x^i))</td>
<td>((t', x'^i))</td>
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<tr>
<td>(Q)</td>
<td>((t + dt, x^i - \beta^i dt))</td>
<td>((t' + \alpha dt, x'^i))</td>
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<tr>
<td>(R)</td>
<td>((t + dt, x^i + dx^i))</td>
<td>((t' + \alpha dt, x'^i + \beta^i dt + dx^i))</td>
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4-velocity = vector \(\vec{PR} / \text{proper time} \tau\)

- in the coordinate frame \(S\): \(u^\mu = \frac{(dt, dx^i)}{d\tau} = \left( \frac{dt}{d\tau}, \frac{dx^i}{d\tau} \right)\)
- in the Eulerian frame \(S'\):

\[
 u'^\mu = \frac{(\alpha dt, \beta^i dt + dx^i)}{dt} = (\alpha u^0, u^i + \beta^i u^0) = \alpha u^0 (1, \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha})
\]
3+1 Eulerian formulation

- Lorentz factor as seen from $S'$ is: $W = -n_\mu u^\mu = \alpha u^0$

- 3-velocity of particle in Eulerian frame: $v^i = \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}$

\[v_i = \gamma_{ij} v^j = \gamma_{ij} \left( \frac{u^j}{\alpha u^0} + \frac{\beta^j}{\alpha} \right) = \frac{1}{\alpha u^0} \gamma_{ij} (u^j + \beta^j u^0) = \frac{1}{\alpha u^0} (\gamma_{ij} u^j + \beta_i u^0) = \frac{1}{\alpha u^0} (g_{i0} u^0 + g_{ij} u^j) = \frac{u_i}{\alpha u^0}\]

- Normalization: 
  \[-1 = g_{\mu\nu} u^\mu u^\nu = -\alpha^2 (u^0)^2 + \gamma_{ij} (u^i + \beta^i u^0) (u^j + \beta^j u^0) = -\alpha^2 (u^0)^2 \left[ 1 - \gamma_{ij} \left( \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha} \right) \left( \frac{u^j}{\alpha u^0} + \frac{\beta^j}{\alpha} \right) \right] = -\alpha^2 (u^0)^2 (1 - \gamma_{ij} v^i v^j)\]

  Lorentz factor: \[W = \alpha u^0 = (1 - \gamma_{ij} v^i v^j)^{1/2}\]
General relativistic hydrodynamics

- GRHD equations consist of the local conservation laws of the **matter current density** and the **energy-momentum** (stress-energy tensor) + fluid equation of state

- **rest mass flux** (proportional to the baryon number flux) is:

\[ J^{\mu} = \rho_0 u^{\mu} \]

- \( \rho_0 = \) rest mass density (baryon number density times average rest mass of the baryons); \( u = \) fluid velocity

- **stress-energy tensor** of an **ideal fluid** (without non-adiabatic processes, s.a., viscosity, magnetic field, radiation)

\[ T^{\mu\nu} = (\rho + p) u^{\mu} u^{\nu} + pg^{\mu\nu} \]

- \( \rho = \) total mass-energy density; \( p = \) pressure
General relativistic hydrodynamics

useful relations: \[ u^0 = \frac{W}{\alpha}, \quad \frac{u^i}{W} = v^i - \frac{\beta^i}{\alpha}, \quad \frac{u_i}{W} = v_i \]

CONSERVATIVE VARIABLES are measured by the Eulerian observers, being defined as:

\[ D = -J^\mu n_\mu = -\rho_0 u^\mu n_\mu = \rho_0 W, \text{ rest mass density} \]
\[ S_j = -T^\mu_\nu n_\mu (\partial_j)^\nu = \alpha T^0_\mu \partial_j = \alpha (\rho + p) u^0 u_j = \]
\[ = (\rho + p) W^2 \frac{u_j}{W} = (\rho + p) W^2 v_j, \text{ momentum density} \]
\[ E = T^\mu_\nu n_\mu n_\nu = \alpha^2 T^{00} = \alpha^2 [ (\rho + p) u^0 u^0 + pg^{00} ] = \]
\[ = (\rho + p) W^2 - \alpha^2 p \frac{1}{\alpha^2} = (\rho + p) W^2 - p, \text{ energy} \]

derived conservative variable: \[ \tau = E - D \]
General relativistic hydrodynamics

\[
T^{00} = \frac{1}{\alpha^2} \quad T^{0i} = \frac{1}{\alpha} (\rho + P) W^2 (v^i - \frac{\beta^i}{\alpha}) + P \frac{\beta^i}{\alpha^2} \\
T^{0i} = \frac{1}{\alpha} S_i \quad T^{ij} = S_j (v^i - \frac{\beta^i}{\alpha}) + P \delta^i_j
\]

● differential form of baryon number conservation (continuity equation) 1st GRHD equation

\[
J^\mu;\mu = \frac{1}{\sqrt{-g}} (\sqrt{-g} J^\mu)_{,\mu} = \frac{1}{\sqrt{-g}} (\sqrt{-g} \rho_0 u^\mu)_{,\mu} = \\
= \frac{1}{\sqrt{-g}} \left( \alpha \sqrt{\gamma} \rho_0 \frac{W}{\alpha} \right)_{,0} + \frac{1}{\sqrt{-g}} \left( \sqrt{-g} \rho_0 W (v^i - \frac{\beta^i}{\alpha}) \right)_{,i} = \\
= \frac{1}{\sqrt{-g}} (\sqrt{\gamma} D)_{,0} + \frac{1}{\sqrt{-g}} D \left( v^i - \frac{\beta^i}{\alpha} \right)_{,i} = 0
\]
General relativistic hydrodynamics

\[(T^\mu_\nu(e_\gamma)^\nu)_\mu = T^\mu_\nu;\mu (e_\gamma)^\nu + T^{\mu\nu}(e_\gamma)_\nu;\mu = T^{\mu\nu} \left((e_\gamma)^\nu,\mu - \Gamma^\lambda_\nu_\mu (e_\gamma)^\lambda\right)\]

For $\gamma = 0$:

\[(T^\mu_\nu(e_0)^\nu)_\mu = (T^{\mu\nu}n_\nu);\mu = (-\alpha T^{\mu0});\mu = -\frac{1}{\sqrt{-g}} \left\{ (\sqrt{-g} \alpha T^{\mu0}),_0 + (\sqrt{-g} \alpha T^{\mu0}),_i\right\} \]

\[= -\frac{1}{\sqrt{-g}} \left\{ \sqrt{\gamma} \alpha^2 T^{\mu0},_0 + \left(\sqrt{-g} \alpha \left[ \frac{1}{\alpha} (\rho + P) W^2 (v^i - \frac{\beta^i}{\alpha}) + p \frac{\beta^i}{\alpha^2} \right] \right),_i\right\} \]

\[= -\frac{1}{\sqrt{-g}} \left\{ \sqrt{\gamma} E,_0 + \left(\sqrt{-g} \left[ E (v^i - \frac{\beta^i}{\alpha}) + pv^i \right] \right),_i\right\} \]

\[T^{\mu\nu} \left((e_0)^\nu,\mu - \Gamma^\lambda_\nu_\mu (e_0)^\lambda\right) = -T^{\mu0} \alpha,\mu + \alpha \Gamma^0_{\nu\mu} T^{\mu\nu} = \alpha \left(-T^{\mu0} (ln \alpha),_\mu + \Gamma^0_{\nu\mu} T^{\mu\nu}\right)\]

\[\frac{1}{\sqrt{-g}} \left\{ \frac{\partial}{\partial t} (\sqrt{\gamma} E) + \frac{\partial}{\partial x^i} \left(\sqrt{-g} \left[ E (v^i - \frac{\beta^i}{\alpha}) + pv^i \right] \right) \right\} = \alpha \left(-T^{\mu0} (ln \alpha),_\mu + \Gamma^0_{\nu\mu} T^{\mu\nu}\right)\]

2nd GRHD equation
For $\gamma = j$:

\[
(T^\mu_\nu(e_j)^\nu)_;\mu = (T^\mu_j)_;\mu = \frac{1}{\sqrt{-g}} \left\{ (\sqrt{-g}T^\mu_j)_,\mu \right\}
= \frac{1}{\sqrt{-g}} \left\{ (\sqrt{-g}T^0_j)_,0 + (\sqrt{-g}T^i_j)_,i \right\}
= - \frac{1}{\sqrt{-g}} \left\{ (\sqrt{-g}S_j)_,0 + (\sqrt{-g} \left[ S_j (v^i - \frac{\beta^i}{\alpha}) + P\delta^i_j \right] )_i \right\}
\]

\[
T^{\mu\nu} \left[ (e_j)_\nu,\mu - \Gamma^\lambda_\nu_\mu (e_j)_\lambda \right] = T^{\mu\nu} \left[ \left[ g_{\nu\lambda} (e_j)^\lambda \right]_\mu,\mu - \Gamma^\lambda_\nu_\mu g_{\lambda\sigma} (e_j)^\sigma \right] = T^{\mu\nu} \left( g_{\nu,j,\mu} - \Gamma^\lambda_\nu_\mu g_{\lambda j} \right)
\]

\[
\frac{1}{\sqrt{-g}} \left\{ \frac{\partial}{\partial t} (\sqrt{\gamma}S_j) + \frac{\partial}{\partial x^i} \left( \sqrt{-g} \left[ S_j (v^i - \frac{\beta^i}{\alpha}) + P\delta^i_j \right] \right) \right\} = T^{\mu\nu} \left( g_{\nu,j,\mu} - \Gamma^\lambda_\nu_\mu g_{\lambda j} \right)
\]

3rd, 4th, and 5th GRHD equations
General relativistic hydrodynamics

GRHD equations can be written in a conservation form as

\[
\frac{1}{\sqrt{-g}} \left\{ \frac{\partial}{\partial t} (\sqrt{\gamma}U) + \frac{\partial}{\partial x^i} (\sqrt{-g}F^i) \right\} = \Sigma
\]

\[
U = \begin{bmatrix} D \\ S_j \\ \tau \end{bmatrix} \quad F^i = \begin{bmatrix} D(v^i - \frac{\beta^i}{\alpha}) \\ S_j(v^i - \frac{\beta^i}{\alpha}) + p\delta^i_j \\ \tau(v^i - \frac{\beta^i}{\alpha}) + pv^i \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0 \\ T^{\mu\nu} (g_{\nu j,\mu} - \Gamma^\lambda_{\nu\mu} g_{\lambda j}) \\ \alpha (T^{\mu 0} (\ln \alpha),_\mu - \Gamma^0_{\mu\nu} T^{\mu\nu}) \end{bmatrix}
\]

- **Conservative variables**
- **Fluxes**
- **Source term**

- For curved spacetime, there exist source terms, arising from the spacetime geometry.
- For Minkowski metric \( \Sigma = 0 \) and \( \sqrt{-g} = \sqrt{\gamma} = 1 \); strict conservation law is possible only in flat spacetime.
IMPORTANT!! Recovering the primitive variables from the conservative ones, \( \mathbf{P} = [\rho_0, v^j, p] \)

- for conservative formulations, the time update of a given numerical algorithm is applied to the conservative variables
- after this update, the vector of the primitive variables must be re-evaluated as those are needed in the Riemann solver
- the relation between the 2 sets of variables is not in closed form and hence, the recovery of the primitive variables is done by using a root-finding procedure (Newton-Raphson scheme)
General relativistic MHD

One have to include: evolution equations for magnetic field (Maxwell equations – divergence free magnetic field and induction equation) + frozen-in condition

\[ T_{\mu\nu} = T_{\mu\nu}^{\text{fluid}} + T_{\mu\nu}^{\text{elmagm}} \]

\[
\begin{bmatrix}
D \\
S_j \\
\tau \\
B^j
\end{bmatrix}
= \begin{bmatrix}
D(v^i - \frac{\beta^i}{\alpha}) \\
S_j(v^i - \frac{\beta^i}{\alpha}) + p\delta^i_j \\
\tau(v^i - \frac{\beta^i}{\alpha}) + pv^i \\
\tilde{v}^i B^j - \tilde{v}^j B^i
\end{bmatrix}
\]

\[
\Sigma = \begin{bmatrix}
0 \\
T_{\mu\nu} (g_{\nu j,\mu} - \Gamma_{\nu\mu}^\lambda g_{\lambda j}) \\
\alpha (T_{\mu 0} (\ln \alpha),\mu - \Gamma_{\mu\nu}^0 T_{\mu\nu}) \\
0
\end{bmatrix}
\]

conservative variables  fluxes  source term

Primitive variables: \( \mathbf{P} = [\rho_0, v^j, p, B^j] \)
Remarks

- GRMHD simulations of jets formation from Kerr BH
  Y. Mizuno, K. Nishikawa, and S. Koide

- Koide et al. (1998) – first 3D GRMHD simulations of jets formation
References


