General Relativistic Magnetohydrodynamics Equations

Ioana Duţan

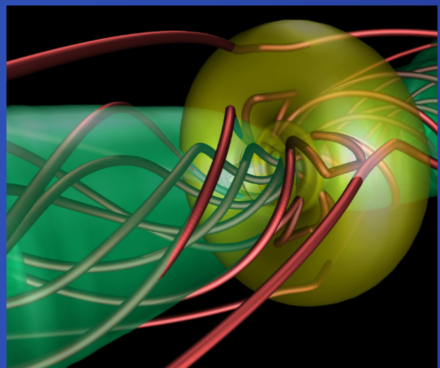
IMPRS Students Workshop

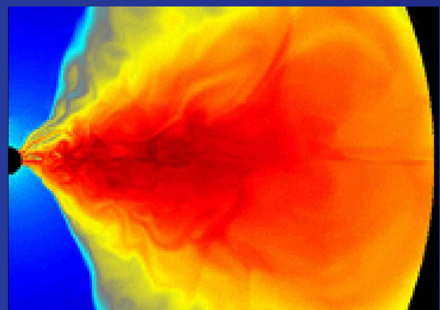
Berggasthof Lansegger, Austria, 28-31 Aug 2006

Introductory remarks

(General) Relativistic (Magneto)hydrodynamics

- a (G)R (M)HD code is used to compute the flow of gas around strong field gravity sources
- used to study supernova collapse and formation of BH, BH-BH binaries, NS-NS binaries, pulsar wind nebulae, accretion disks, relativistic jets from AGN and microquasars, etc.





Introductory remarks

- **Propose of this material**
  - to introduce or review some aspects of GR
  - to introduce the 3+1 decomposition of the spacetime (in the Eulerian formulation)
  - to provide a small derivation of the conservative systems of the hyperbolic PDE of the GRHD

- **Notes on the level of this material**
  - if it is too easy, just treat it as review, perhaps from a different perspective
  - if it is too fuzzy, concentrate on the concepts...

General relativity background

- metric element:

$$ds^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu$$

- for a flat spacetime of special relativity:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

  - $ds^2 < 0$, interval is timelike; $ds^2 > 0$, interval is spacelike; $ds^2 = 0$, it is null
- 1D curve $x^\mu(\lambda)$ in spacetime describes a series of events
- timelike curve (worldline) is parameterized by the proper time τ

General relativity background

- 4-velocity is defined as: $u^\mu = \frac{dx^\mu(\tau)}{d\tau}$
- in SR, the 4-velocity components are

$$u^\mu = (u^0, u^1, u^2, u^3) = (W, W\vec{v}), \text{ where } W = \frac{1}{\sqrt{1-\vec{v}^2}}$$

- imagine 2 particles with worldlines that meet at point A, having 4-velocity u_μ and v^μ; their product is an invariant (it can be evaluated in an arbitrary reference frame)(Fig. 1)
- in particular, in the Lorentz reference frame comoving with u^μ you have

$$u'_\mu = (-1, 0, 0, 0), \text{ and then } u_\mu v^\mu = u'_\mu v'^\mu = -v'^0 = -W$$

General relativity background

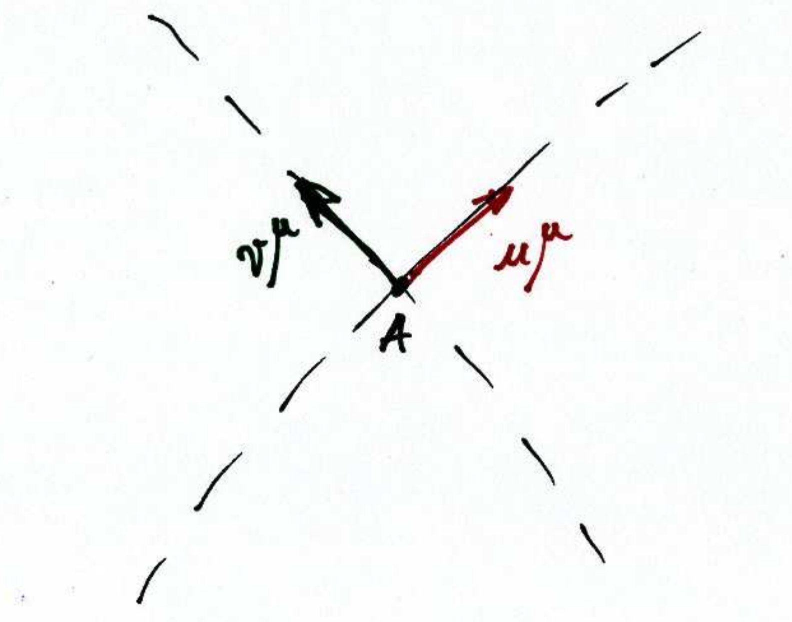


Fig 1. Worldlines intersection

3+1 Eulerian formulation

3+1 decomposition

- spacetime is foliated into a set of non-intersecting spacelike hypersurfaces, parameterized by a parameter usually called time t, s.t., the evolution between these surfaces is described by two kinematic variables
- metric can be written in a particular way

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- which in a component form is

$$\begin{Vmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{Vmatrix} = \begin{Vmatrix} \beta_s \beta^s - \alpha^2 & \beta_j \\ \beta_i & \gamma_{ij} \end{Vmatrix}, \quad \text{where } \boxed{\gamma_{ij} = g_{ij}} \; i, j = 1, 2, 3$$

$\gamma_{ij} =$ 3-metric induced on each spacelike slice

3+1 Eulerian formulation

- $\gamma_{ik}\gamma^{kj} = \delta_i^j$; gymnastics of indices $\beta_i = \gamma_{ij}\beta^j$
- decomposition of the volume associated with the 4-metric into the volume associated with the 3-metric

$$\boxed{\sqrt{-g} = \alpha\sqrt{\gamma}}, \qquad g = det(g_{\mu\nu}),\ \gamma = det(\gamma_{ij})$$

- let's consider 2 close spacelike hypersuperfaces $\Sigma(t)$ and $\Sigma(t+dt)$ (Fig. 2)
- lapse function α describes the rate of advance of time along a timelike unit vector normal to the hypersurface
- spacelike shift vectors β^i describe the motion of coordinates within a surface

3+1 Eulerian formulation

- 4-vector $\mathbf{n}^\mu$ is the unit normal vector to the $\Sigma(t)$

$$\mathbf{n}_\mu = (-\alpha, 0, 0, 0), \qquad \mathbf{n}^\mu = (\frac{1}{\alpha}, \frac{-\beta^i}{\alpha})$$

- Eulerian observers are observers having $\mathbf{n}$ as 4-velocity, at rest in the slice $\Sigma(t)$ and moving $\perp$ to this slice with clocks showing proper time
- i.e., the basis adapted to the Eulerian observer frames is:

$$\mathbf{e}_{(\mu)} = \{\mathbf{n}, \partial_i\}$$

- 4-vector u^μ is the 4-velocity of some particle

3+1 Eulerian formulation

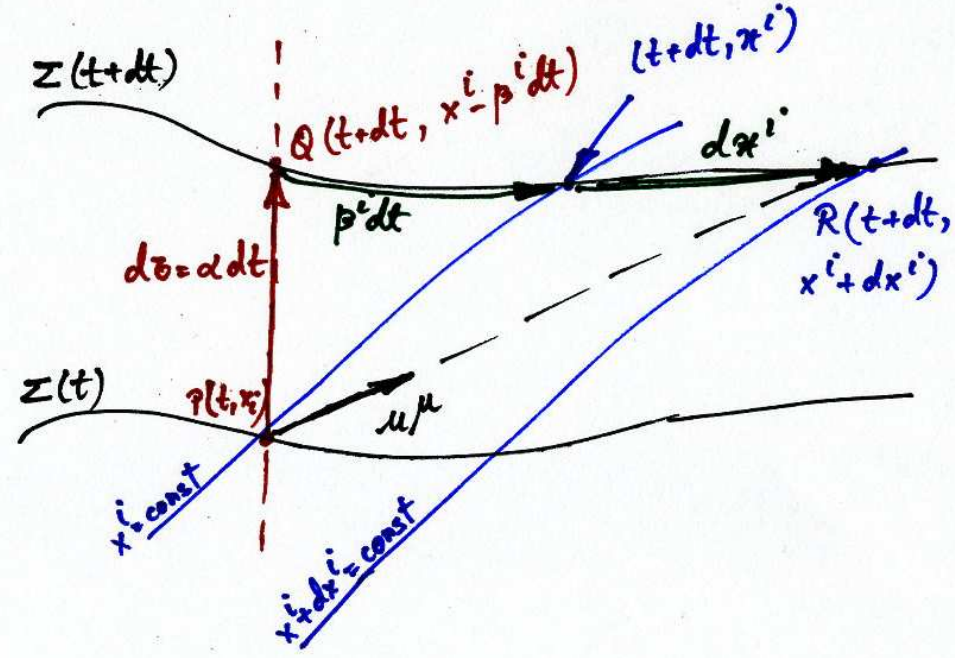


Fig 2. Geometrical interpretation of α, β^i.

3+1 Eulerian formulation

- Let's translate the 4-velocity of a particle from the arbitrary coordinate frame (S) to the Eulerian frame (S')

	S	S'
P	(t, x^i)	$(t', {x'}^i)$
Q	$(t+dt, x^i - \beta^i dt)$	$(t' + \alpha dt, {x'}^i)$
R	$(t+dt, x^i + dx^i)$	$(t' + \alpha dt, {x'}^i + \beta^i dt + dx^i)$

4-velocity = vector $\vec{PR}$ / proper time τ

- in the coordinate frame S: $u^\mu = \frac{(dt, dx^i)}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right)$
- in the Eulerian frame S':

$$u'^\mu = \frac{(\alpha dt, \beta^i dt + dx^i)}{dt} = (\alpha u^0, u^i + \beta^i u^0) = \alpha u^0 (1, \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha})$$

3+1 Eulerian formulation

- Lorentz factor as seen from S' is: $W = -n_\mu u^\mu = \alpha u^0$
- 3-velocity of particle in Eulerian frame: $v^i = \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}$

$$v_i = \gamma_{ij} v^j = \gamma_{ij}\left(\frac{u^j}{\alpha u^0} + \frac{\beta^j}{\alpha}\right) = \frac{1}{\alpha u^0}\gamma_{ij}(u^j + \beta^j u^0) =$$

$$= \frac{1}{\alpha u^0}(\gamma_{ij} u^j + \beta_i u^0) = \frac{1}{\alpha u^0}(g_{i0}u^0 + g_{ij}u^j) = \frac{u_i}{\alpha u^0}$$

- Normalization: $-1 = g_{\mu\nu}u^\mu u^\nu = -\alpha^2 (u^0)^2 + \gamma_{ij}(u^i + \beta^i u^0)(u^j + \beta^j u^0)$=

$$= -\alpha^2 (u^0)^2 \left[1 - \gamma_{ij}\left(\frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}\right)\left(\frac{u^j}{\alpha u^0} + \frac{\beta^j}{\alpha}\right)\right] = -\alpha^2 (u^0)^2 (1 - \gamma_{ij} v^i v^j)$$

Lorentz factor: $W = \alpha u^0 = (1 - \gamma_{ij} v^i v^j)^{1/2}$

General relativistic hydrodynamics

- GRHD equations consist of the local conservation laws of the matter current density and the energy-momentum (stress-energy tensor) + fluid equation of state
- rest mass flux (proportional to the baryon number flux) is:

$$\boxed{J^\mu = \rho_0 u^\mu}$$

$\rho_0 =$ rest mass density (baryon number density times average rest mass of the baryons); $\mathbf{u} =$ fluid velocity

- stress-energy tensor of an ideal fluid (without non-adiabatic processes, s.a., viscosity, magnetic field, radiation)

$$\boxed{T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}}$$

$\rho =$ total mass-energy density; $p =$ pressure

General relativistic hydrodynamics

- useful relations: $u^0 = \frac{W}{\alpha}, \quad \frac{u^i}{W} = v^i - \frac{\beta^i}{\alpha}, \quad \frac{u_i}{W} = v_i$
- **CONSERVATIVE VARIABLES** are measured by the Eulerian observers, being defined as:

$$D = -J^\mu n_\mu = -\rho_0 u^\mu n_\mu = \rho_0 W, \; rest - mass\ density$$

$$S_j = -T^\mu{}_\nu n_\mu (\partial_j)^\mu = \alpha T^0{}_j = \alpha(\rho + p) u^0 u_j =$$

$$= (\rho + p) W^2 \frac{u_j}{W} = (\rho + p) W^2 v_j, \; momentum\ density$$

$$E = T^{\mu\nu} n_\mu n_\nu = \alpha^2 T^{00} = \alpha^2 [(\rho + p) u^0 u^0 + p g^{00}] =$$

$$= (\rho + p) W^2 - \alpha^2 p \frac{1}{\alpha^2} = (\rho + p) W^2 - p, \; energy$$

  - derived conservative variable: $\tau = E - D$

General relativistic hydrodynamics

$$T^{00} = \frac{1}{\alpha^2} \qquad T^{0i} = \frac{1}{\alpha}(\rho + P)W^2(v^i - \frac{\beta^i}{\alpha}) + P\frac{\beta^i}{\alpha^2}$$

$$T^0{}_i = \frac{1}{\alpha}S_i \qquad T^i{}_j = S_j(v^i - \frac{\beta^i}{\alpha}) + P\delta^i_j$$

- differential form of baryon number conservation (continuity equation) 1st GRHD equation

$$\begin{aligned} J^\mu{}_{;\mu} &= \frac{1}{\sqrt{-g}}(\sqrt{-g}J^\mu)_{,\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\rho_0 u^\mu)_{,\mu} = \\ &= \frac{1}{\sqrt{-g}}\left(\alpha\sqrt{\gamma}\rho_0\frac{W}{\alpha}\right)_{,0} + \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\rho_0 W(v^i - \frac{\beta^i}{\alpha})\right)_{,i} = \\ &= \frac{1}{\sqrt{-g}}(\sqrt{\gamma}D)_{,0} + \frac{1}{\sqrt{-g}}D\left(v^i - \frac{\beta^i}{\alpha}\right)_{,i} = 0 \end{aligned}$$

General relativistic hydrodynamics

$$(T^{\mu}{}_{\nu}(\mathbf{e}_\gamma)^{\nu})_{;\mu} = T^{\mu}{}_{\nu;\mu}(\mathbf{e}_\gamma)^{\nu} + T^{\mu\nu}(\mathbf{e}_\gamma)_{\nu;\mu} = T^{\mu\nu}\left((\mathbf{e}_\gamma)_{\nu,\mu} - \Gamma^{\lambda}_{\nu\mu}(\mathbf{e}_\gamma)_{\lambda}\right)$$

For $\gamma = 0$:

$$(T^{\mu}{}_{\nu}(\mathbf{e}_0)^{\nu})_{;\mu} = (T^{\mu\nu}n_\nu)_{;\mu} = (-\alpha T^{\mu 0})_{;\mu} = -\frac{1}{\sqrt{-g}}\left\{(\sqrt{-g}\alpha T^{00})_{,0} + (\sqrt{-g}\alpha T^{i0})_{,i}\right\}$$

$$= -\frac{1}{\sqrt{-g}}\left\{(\sqrt{\gamma}\alpha^2 T^{00})_{,0} + \left(\sqrt{-g}\alpha\left[\frac{1}{\alpha}(\rho+P)W^2(v^i - \frac{\beta^i}{\alpha}) + p\frac{\beta^i}{\alpha^2}\right]\right)_{,i}\right\}$$

$$= -\frac{1}{\sqrt{-g}}\left\{(\sqrt{\gamma}E)_{,0} + \left(\sqrt{-g}\left[E(v^i - \frac{\beta^i}{\alpha}) + pv^i\right]\right)_{,i}\right\}$$

$$T^{\mu\nu}\left((\mathbf{e}_0)_{\nu,\mu} - \Gamma^{\lambda}_{\nu\mu}(\mathbf{e}_0)_{\lambda}\right) = -T^{\mu 0}\alpha_{,\mu} + \alpha\Gamma^{0}_{\nu\mu}T^{\mu\nu} = \alpha\left(-T^{\mu 0}(\ln\alpha)_{,\mu} + \Gamma^{0}_{\nu\mu}T^{\mu\nu}\right)$$

$$\boxed{\frac{1}{\sqrt{-g}}\left\{\frac{\partial}{\partial t}(\sqrt{\gamma}E) + \frac{\partial}{\partial x^i}\left(\sqrt{-g}\left[E(v^i - \frac{\beta^i}{\alpha}) + pv^i\right]\right)\right\} = \alpha\left(-T^{\mu 0}(\ln\alpha)_{,\mu} + \Gamma^{0}_{\nu\mu}T^{\mu\nu}\right)}$$

2nd GRHD equation

General relativistic hydrodynamics

For $\gamma = j$:

$$(T^{\mu}{}_{\nu}(\mathbf{e}_j)^{\nu})_{;\mu} = (T^{\mu}{}_{j})_{;\mu} = \frac{1}{\sqrt{-g}}\left\{\left(\sqrt{-g}T^{\mu}{}_{j}\right)_{,\mu}\right\}$$

$$= \frac{1}{\sqrt{-g}}\{(\sqrt{-g}T^{0}{}_{j})_{,0} + (\sqrt{-g}T^{i}{}_{j})_{,i}\}$$

$$= -\frac{1}{\sqrt{-g}}\left\{(\sqrt{-\gamma}S_j)_{,0} + \left(\sqrt{-g}\left[S_j(v^i - \frac{\beta^i}{\alpha}) + p\delta^i_j\right]\right)_{,i}\right\}$$

$$T^{\mu\nu}\left((\mathbf{e}_j)_{\nu,\mu} - \Gamma^{\lambda}_{\nu\mu}(\mathbf{e}_j)_{\lambda}\right) = T^{\mu\nu}\left(\left[g_{\nu\lambda}(\mathbf{e}_j)^{\lambda}\right]_{,\mu} - \Gamma^{\lambda}_{\nu\mu}g_{\lambda\sigma}(\mathbf{e}_j)^{\sigma}\right) = T^{\mu\nu}\left(g_{\nu j,\mu} - \Gamma^{\lambda}_{\nu\mu}g_{\lambda j}\right)$$

$$\boxed{\frac{1}{\sqrt{-g}}\left\{\frac{\partial}{\partial t}(\sqrt{\gamma}S_j) + \frac{\partial}{\partial x^i}\left(\sqrt{-g}\left[S_j(v^i - \frac{\beta^i}{\alpha}) + P\delta^i_j\right]\right)\right\} = T^{\mu\nu}\left(g_{\nu j,\mu} - \Gamma^{\lambda}_{\nu\mu}g_{\lambda j}\right)}$$

3rd, 4th, and 5th GRHD equations

General relativistic hydrodynamics

GRHD equations can be written in a conservation form as

$$\frac{1}{\sqrt{-g}}\left\{\frac{\partial}{\partial t}(\sqrt{\gamma}\mathbf{U})+\frac{\partial}{\partial x^i}(\sqrt{-g}\mathbf{F}^i)\right\}=\mathbf{\Sigma}$$

$$\mathbf{U}=\begin{bmatrix} D \\ S_j \\ \tau \end{bmatrix} \qquad \mathbf{F}^i=\begin{bmatrix} D(v^i-\frac{\beta^i}{\alpha}) \\ S_j(v^i-\frac{\beta^i}{\alpha})+p\delta^i_j \\ \tau(v^i-\frac{\beta^i}{\alpha})+pv^i \end{bmatrix} \qquad \mathbf{\Sigma}=\begin{bmatrix} 0 \\ T^{\mu\nu}\left(g_{\nu j,\mu}-\Gamma^\lambda_{\nu\mu}g_{\lambda j}\right) \\ \alpha\left(T^{\mu 0}(\ln\alpha)_{,\mu}-\Gamma^0_{\mu\nu}T^{\mu\nu}\right) \end{bmatrix}$$

conservative variables *fluxes* *source term*

- for curved spacetime, there exist source terms, arising from the spacetime geometry
- for Minkowski metric $\Sigma = 0$ and $\sqrt{-g}=\sqrt{\gamma}=1$; strict conservation low is possible only in flat spacetime

General relativistic hydrodynamics

IMPORTANT!! Recovering the primitive variables from the conservative ones, $\mathbf{P} = [\rho_0, v^j, p]$

- for conservative formulations, the time update of a given numerical algorithm is applied to the conservative variables
- after this update, the vector of the primitive variables must be re-evaluated as those are needed in the Riemann solver
- the relation between the 2 sets of variables is not in closed form and hence, the recovery of the primitive variables is done by using a root-finding procedure (Newton-Raphson scheme)

General relativistic MHD

One have to include: evolution equations for magnetic field (Maxwell equations – divergence free magnetic field and induction equation) + frozen-in condition

$$T^{\mu\nu} = T^{\mu\nu}{}_{fluid} + T^{\mu\nu}{}_{elmagm}$$

$$\mathbf{U} = \begin{bmatrix} D \\ S_j \\ \tau \\ B^j \end{bmatrix} \qquad \mathbf{F}^i = \begin{bmatrix} D(v^i - \frac{\beta^i}{\alpha}) \\ S_j(v^i - \frac{\beta^i}{\alpha}) + p\delta^i_j \\ \tau(v^i - \frac{\beta^i}{\alpha}) + pv^i \\ \tilde{v}^i B^j - \tilde{v}^j B^i \end{bmatrix} \qquad \mathbf{\Sigma} = \begin{bmatrix} 0 \\ T^{\mu\nu}\left(g_{\nu j,\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\lambda j}\right) \\ \alpha\left(T^{\mu 0}(\ln\alpha)_{,\mu} - \Gamma^{0}_{\mu\nu} T^{\mu\nu}\right) \\ 0 \end{bmatrix}$$

conservative variables *fluxes* *source term*

Primitive variables: $\mathbf{P} = \left[\rho_0, v^j, p, B^j\right]$

Remarks

- GRMHD simulations of jets formation from Kerr BH
  Y. Mizuno, K. Nishikawa, and S. Koide
- Koide et al. (1998) – first 3D GRMHD simulations of jets formation

References

1. Font, J. A. – *Numerical Hydrodynamics in General Relativity*, http://www.livingreviews.org/lrr-2003-4
2. Font, J. A. – *General relativistic hydrodynamics and magnetohydrodynamics and their applications*, Plasma Phys. Control. Fussion **47**, B679-B690, 2005
3. Koide, S. – *Magnetic extraction of black hole rotational energy: Method and results of general relativistic magnetohydrodynamic simulations in Kerr space-time*, Phys. Rev. D, vol. 67, Issue 10, 2003
4. Mizuno, Y. et al. – *RAISHIN: A High-Resolution Three-Dimensional General Relativistic Magnetohydrodynamics Code*, astro-ph/0609004